RWTH Aachen University
Chair i7 / Logic and Theory of Discrete Systems
Computer Science (M.Sc.)

Homomorphism Indistinguishability Relations induced by Quantum Groups

Master Thesis
Submitted for the Degree of
Master of Science

Submitted by
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on November 5, 2024

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Abstract

Homomorphism indistinguishability is a way of characterising many natural equivalence relations on graphs. Two graphs G and H are called homomorphism indistinguishable over a graph class $\mathcal{F}$ if for each $F \in \mathcal{F}$, the number of homomorphisms from F to G equals the number of homomorphisms from F to H. Examples of such equivalence relations include isomorphism and cospectrality, as well as equivalence with respect to many formal logics.

Quantum groups are a generalisation of topological groups that describe "non-commutative symmetries" and, inter alia, have applications in quantum information theory. An important subclass are the easy quantum groups, which enjoy a combinatorial characterisation and have been fully classified by Raum and Weber.

A recent connection between these seemingly distant concepts was made by Mančinska and Roberson, who showed that quantum isomorphism, a relaxation of classical isomorphism that can be phrased in terms of the quantum symmetric group, is equivalent to homomorphism indistinguishability over the class of planar graphs. Their techniques suggest that analogous results may hold for other quantum groups as well. We give an intuitive overview of the theory of compact quantum groups, and proceed to prove a generalisation of Mančinska and Roberson's result for all (orthogonal) easy quantum groups. This includes a full classification of the intertwiner graphs associated to each easy quantum group.

Acknowledgements

I want to use this space to thank a few people that had a profound influence on me, not only during the process of writing this thesis, but during my entire time at RWTH.

First of all, I would like to express my sincere gratitude to Professor Martin Grohe for encouraging me to expand on and publish the results of my Bachelor's thesis. The resulting paper marked one of the first steps in my research journey, and the process was deeply instructive. Furthermore, I want to express my appreciation to Professor Erich Grädel, whose undergraduate lectures on mathematical logic first made me excited about the mathematical foundations of computer science, and whose excellent lecture notes majorly influenced my writing. My appreciation also goes to Professor Maarten Wegewijs for his great lectures on quantum information theory and his captivating choice of topics. His enthusiasm is contagious and his course "Quantum Information" immediately sparked my interest in the subject.

These acknowledgements would not be complete without thanking Tim Seppelt, who supervised this thesis. He introduced me to quantum graph theory, which combines my two main research interests and will be the topic of my PhD. I also thorougly enjoyed our conversations and the joint work on my first paper together with Dr. Eva Fluck.

Finally, I must thank my mom, who always had my back. Without her I would not be where I am today (and not just in the literal sense).

I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I-
I took the one less traveled by,
And that has made all the difference.

Robert Frost, The Road Not Taken

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1 Introduction

Homomorphism counts capture graph structure. More precisely, two graphs G and H are said to be *homomorphism indistinguishable* over a class of graphs $\mathcal{F}$ if for all graphs $F \in \mathcal{F}$, the number of homomorphisms from F to G equals the number of homomorphisms from F to H. In 1967, Lovász [18] first showed that isomorphism is equivalent to homomorphism indistinguishability over all graphs. Since then, a wide spectrum of equivalence relations have been characterised as homomorphism indistinguishability relations. Examples include natural equivalence relations such as cospectrality and having the same degree sequence, as well as equivalences in different formal logics [10, 13, 11], results in category theory [7, 23], and characterisations of different linear programming relaxations [9, 27].

Particularly interesting is a recent result by Mančinska and Roberson [22], who showed that *quantum isomorphism* is equivalent to homomorphism indistinguishability over the class of planar graphs. Quantum isomorphism was originally defined in [2] in terms of the *graph isomorphism game*. This is a non-local game in which two players have to convince a verifier that two graphs are isomorphic. They are not allowed to communicate, but may agree on a strategy beforehand. Classically, the players have a winning strategy if and only if the graphs are in fact isomorphic. On the other hand, we call the two graphs *quantum isomorphic* if the players have a winning strategy when being allowed to coordinate their answers by measuring a shared entangled quantum state.[1]

Results by Lupini, Mančinska, and Roberson [19] have shown that classical and quantum and isomorphism share very deep similarities. Classical isomorphism of graphs G and H can be equivalently phrased as the existence of a permutation matrix P such that $PA_G = A_H P$, where A_G, A_H are the adjacency matrices of G and H respectively. Slightly less obvious is that if G and H are connected, then they are isomorphic if and only if there exists an orbit of the automorphism group of the disjoint union $G \cup H$, that intersects both G and H non-trivially. As such, classical isomorphism is closely related to the symmetric group S_n and automorphism groups. It turns out that the same picture holds for quantum isomorphism, where permutation matrices are replaced by so called *quantum permutation matrices*—a generalisation of permutation matrices with entries in some C*-algebra—and where two connected graphs are quantum isomorphic if and only if there exists an orbit of the *quantum automorphism group* of $G \cup H$ that intersects G and H non-trivially. Hence quantum isomorphism is closely related to the *quantum symmetric group* S_n^+ and quantum automorphism groups $\mathrm{Qut}(\cdot)$.

Quantum groups like the quantum symmetric group generalise classical groups and can be seen as describing "non-commutative symmetries". They have their origins in mathematical physics and have been extensively studied in the last 30 years. In their proof, Mančinska and Roberson showed that the *intertwiners*—an important concept in the representation theory of quantum groups—of (*orthogonal*) *easy quantum groups* can be seen as *homomorphism tensors*, which are matrices that keep track of homomorphism counts from bilabelled graphs. Through a Tannaka-Krein-type duality originally due to Woronowicz [34], the bilabelled graphs in question entirely determine the easy quantum group. For the quantum symmetric group these graphs are shown to be planar bilabelled graphs, which eventually yields the equivalence of quantum

[1] Technically, there are multiple non-equivalent definitions of measuring a shared state, which all give rise to different notions of quantum isomorphism. In fact, the definition most often called "quantum isomorphism" in the literature, the *tensor product framework*, is not the notion we are interested in (the *commuting operator framework*). However, given the parallels between classical isomorphism and quantum isomorphism in the commuting operator framework, our nomenclature seems the most natural.

isomorphism and homomorphism indistinguishability over planar graphs.

In light of these connections, a natural question is whether similar homomorphism indistinguishability results can be obtained for other easy quantum groups, which have, in fact, been fully classified by Raum and Weber [26]. In this thesis, we aim to eventually answer this question. We will start, however, at the very beginning. After going over some preliminaries in Chapter 2, we will introduce the theory of compact quantum groups in Chapter 3. We establish the mathematical foundations and formally define compact quantum groups. We will then take a look at how concepts from classical group theory, such as orbits and representation theory, can be generalised to the quantum case. The chapter culminates with an overview of the full classification of easy quantum groups.

In Chapter 4, we then retrace Mančinska and Roberson's proof of the homomorphism indistinguishability characterisation of quantum isomorphism, formally introducing the connection between homomorphism counts and the representation theory of easy quantum groups in the process.

Chapter 5 is dedicated to answering the question whether analogous characterisations can be obtained for the other easy quantum groups. We answer the question affirmatively: We introduce the *graph instantiations* of an easy quantum group Q, which are a generalisation of quantum automorphism groups, and define the *intertwiner graphs*, which are intimately related to the bilabelled graphs determining Q. We then prove that two graphs G and H are homomorphism indistinguishable over the unlabelled intertwiner graphs of Q if and only if the system of equations $\mathcal{U}A_G = A_H\mathcal{U}$ has a solution $\mathcal{U}$ over some C*-algebra that satisfies certain additional constraints based on the intertwiners of Q.

Theorem 5.22 (Informal)**.** *Let Q be an easy quantum group and let $\mathcal{F}$ be the set of its intertwiner graphs. Then for all graphs G and H, the following are equivalent.*

(*i*) *G and H are homomorphism indistinguishable over the unlabelled graphs in $\mathcal{F}$.*

(*ii*) *The graph instantiations $Q(G)$ and $Q(H)$ have similar intertwiner spaces.*

(*iii*) *There exists a matrix $\mathcal{U}$ with entries in some C*-algebra, such that $\mathcal{U}A_G = A_H\mathcal{U}$ and $\mathcal{U}^{\otimes k}M = M\mathcal{U}^{\otimes \ell}$ for all intertwiners M of Q and suitable $k, \ell \in \mathbb{N}$.*

We will see that the constraints imposed on $\mathcal{U}$ by the intertwiners of Q correspond to very natural properties of matrices, such as orthogonality or restrictions on row and column sums. Based on Raum and Weber's classification, we moreover identify the classes of unlabelled intertwiner graphs of all easy quantum groups. We find that for every easy quantum group, these graph classes are either all graphs, all planar graphs, all paths and cycles, or all cycles. This yields several concrete corollaries of our theorem and partially answers a question by Mančinska and Roberson. We conclude with Chapter 6, where we discuss our results and point out some natural further questions.

This thesis is written to be as self-contained as possible. We only assume basic knowledge of linear algebra and discrete mathematics. Readers familiar with general topology and/or basic category theory may skip the introductory Sections 2.2 and 2.3 without loss of continuity.

2 Preliminaries

2.1 Notation and Basic Definitions

General Notation. We take the natural numbers $\mathbb{N}$ to include 0. Nevertheless, for $n \in \mathbb{N}$ we write $[n]$ for the set $\{1, \dots, n\}$, and let $[0] = \varnothing$. The *restriction* of a function $f\colon A \to C$ to some set $B \subseteq A$ is the function $f|_B\colon B \to C$ with $f|_B(x) = f(x)$ for all $x \in B$. For functions $f\colon A \to C$, $g\colon B \to C$ where A, B are disjoint, we write $f \cup g$ for the union of f and g, that is, the function mapping x to $f(x)$ if $x \in A$ and to $g(x)$ if $x \in B$. We denote tuples by bold lowercase letters. For tuples $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_m)$, we let $\mathbf{xy} \coloneqq (x_1, \dots, x_n, y_1, y_m)$. We also write (x^n) for the tuple of length n in which every entry is x.

We will define and use all kinds of subobject relations, which we all denote using the symbols $\subseteq, \subsetneq$. By convention, the symbols always refer to the strongest available notion, which will always be clear from context. For example, if A and B are sets, then $A \subseteq B$ means that A is a subset of B; if A and B moreover enjoy (distinct) group structures, we might write $A \nsubseteq B$ to denote that A is not a subgroup of B, even though $A \subseteq B$ as sets.

Linear Algebra. We denote the $n \times n$ identity matrix by I_n. If the dimensions are clear from context, we omit the subscript. For a complex matrix $M \in \mathbb{C}^{n \times m}$, we denote its transpose by M^T, its conjugate transpose by M^* and its entrywise conjugate by $\overline{M}$. We write $\operatorname{diag}(x_1, \dots, x_n)$ for the $n \times n$ diagonal matrix with entries $x_1, \dots, x_n$. The *sum-of-entries* function is defined as $\operatorname{soe}(M) = \sum_{ij} M_{ij}$. We will often need the *tensor product* $A \otimes B$ of linear maps A, B, which can be identified with the *Kronecker product* of matrices, i.e. the matrix with entries $(A \otimes B)_{i_1 i_2, j_1 j_2} = A_{i_1 j_1} \cdot B_{i_2 j_2}$. We write $M^{\otimes k}$ for the k-fold tensor product $M \otimes \cdots \otimes M$, and let $M^{\otimes 0} = (1)$.

By $O_n \subseteq \mathbb{C}^{n \times n}$ we denote the set of all $n \times n$ orthogonal matrices, i.e. all real matrices $M \in \mathbb{C}^{n \times n}$ with $M^\mathsf{T} M = M M^\mathsf{T} = I$. The set of all $n \times n$ unitary matrices, i.e. all $M \in \mathbb{C}^{n \times n}$ with $M^* M = M M^* = I$, we denote by U_n. We call a matrix *bistochastic* if it is orthogonal and has row- and column sum 1. Compare this with *doubly-stochastic* matrices, which are not required to be orthogonal but need to have non-negative entries.

Structures. A *structure* is a set A equipped with relations $R_1, R_2, \dots$ and functions $f_1, f_2, \dots$, where $R_i \subseteq A^{\operatorname{ar}(R_i)}$, $f_i \coloneqq A^{\operatorname{ar}(f_i)} \to A$, and $\operatorname{ar}(\cdot) \in \mathbb{N}$ is called the *arity*. Examples are groups $(G, \cdot)$, rings $(R, +, \cdot, 1)$, and fields $(F, +, \cdot)$. We can also view vector spaces over a field $\mathbb{K}$ as structures $(V, +, (\cdot k)_{k \in \mathbb{K}})$, where $\cdot k\colon V \to V$ represents scalar multiplication by k.

Graphs. A *graph* G is a set $V(G)$ of vertices equipped with a binary relation $E(G) \subseteq V(G)^2$ defining the edges. Unless otherwise specified, we assume all graphs to be *simple*: they are undirected and do not contain loops, that is, $E(G)$ is irreflexive and symmetric. We usually write uv or vu to denote the edge $(u, v) = (v, u) \in E(G)$. We can define a matrix $A \in \{0,1\}^{V(G) \times V(G)}$ with $A_{uv} = 1 \Leftrightarrow uv \in E(G)$, which is called the *adjacency matrix* of G. The class of all graphs is denoted by $\mathcal{G}$.

We call H a *subgraph* of G if H can be obtained from G by removing vertices and edges. A graph is *connected* if there exists a path between any two vertices. The *disjoint union* $G \cup H$ of G and H is the graph given by $(V(G) \sqcup V(H), E(G) \sqcup E(H))$, where $\sqcup$ denotes the disjoint union of sets. We *subdivide* an edge $uv \in E(G)$ by removing it, adding a new vertex w and adding the edges uw and wv. If $v_1, \dots, v_k$ is a path in G such that v_i has degree 2 for all $1 < i < k$, we

unsubdivide it by removing the inner vertices and adding the edge v_1v_k. We *contract* an edge $uv \in E(G)$ by removing it, identifying u and v, and suppressing any resulting multiple edges.

A *planar embedding* of a graph G consists of an injective function $f\colon V(G) \to \mathbb{R}^2$ and a set of curves $h_{uv}[0,1] \to \mathbb{R}^2$ for all $uv \in E(G)$ such that $h_{uv}(0) = f(u)$ and $h_{uv}(1) = f(v)$, and for distinct edges $e, e' \in E(G)$ we have $h_e(x) \neq h_{e'}(y)$ except for $x, y \in \{0,1\}$. A graph is called *planar* if it has a planar embedding, or informally if it can be drawn without any edges crossing. The *faces* of planar embedding are the connected regions of $\mathbb{R}^2 \setminus \bigcup_{e\in E(G)} \operatorname{img} h_e$. A *facial cycle* is a cycle in G that is the boundary of a face in some planar embedding.

Convention. *For simplicity, we will assume that $V(G) = [|V(G)|]$. This means that the class of all graphs is a set, and that the adjacency matrix of a graph uniquely corresponds to a matrix $\mathbb{C}^{|V(G)|\times|V(G)|}$. Note that this is done without loss of generality, since every graph is isomorphic to a graph of this form and all our results will be isomorphism-invariant.*

Mappings & Morphisms. A *homomorphism* is a structure-preserving mapping. More precisely, if $(A, R_1^A, \dots, f_1^B, \dots)$ and $(B, R_1^B, \dots, f_1^B, \dots)$ are two structures with relations $R_i^X \subseteq X^{r_i}$ and and functions $f_i^X\colon X^{s_i} \to X$ for $X \in \{A, B\}$ and $i, r_i, s_i \in \mathbb{N}$, a homomorphism from A to B is a mapping $h\colon A \to B$ such that $h(f_i^A(x_1, \dots, x_{s_i})) = f_i^B(h(x_1), \dots, h(x_{s_i}))$ and $(x_1, \dots, x_{r_i}) \in R_i^A \implies (h(x_1), \dots, h(x_{r_i})) \in R_i^B$ for all i. We call h an *isomorphism* if it is bijective and additionally $R_i^A x_1 \dots x_{r_i} \iff R_i^B h(x_1) \dots h(x_{r_i})$. An *automorphism* is an isomorphism from a structure to itself. We call $h\colon A \to B$ an *antihomomorphism* if it reverses the order of function arguments, that is, $h(f_i^A(x_1, \dots, x_{s_i})) = f_i^B(h(x_{s_i}), \dots, h(x_1))$. *Antiisomorphisms* and *antiautomorphisms* are defined analogously. A map $h\colon A \to A$ is called an *involution* or *involutive* if $h(h(x)) = x$ for all $x \in A$.

For graphs G, H, a *graph homomorphism* from G to H is thus a mapping $h\colon V(G) \to V(H)$ that satisfies $uv \in E(G) \implies h(u)h(v) \in E(H)$. A *vector space homomorphism* is just a linear map. For any graph F, we denote by $\operatorname{Hom}(F, G)$ the set of homomorphisms from F to G, and let $\hom(F, G) = |\operatorname{Hom}(F, G)|$. We then say that G and H are *homomorphism indistinguishable* over a class of graphs $\mathcal{F} \subseteq \mathcal{G}$ if $\hom(F, G) = \hom(F, H)$ for all $F \in \mathcal{F}$.

2.2 A Crash Course in Topology

Quantum groups as we will introduce them in the next chapter are fundamentally topological objects. For a thorough treatment of (general) topology refer to [24]. However, for the reader's convenience, we want to give a brief overview over the central notions and definitions that we will use.

The main idea of general topology is to endow sets with a notion of *closeness*. The most intuitive way of doing this is by assigning each elemement a collection of *neighbourhoods*. In the following, let X be a set. Corresponding to the idea of closeness, we call the elements of X *points*.

Definition 2.1. For each $x \in X$, the *neighbourhoods* of x are given by the non-empty set $\mathcal{N}(x) \subseteq 2^X$, which satisfies the following properties.

- If $N \in \mathcal{N}(x)$ then $x \in N$.
- If $N, M \in \mathcal{N}(x)$ then $N \cap M \in \mathcal{N}(x)$.
- If $N \in \mathcal{N}(x)$ and $N \subseteq M \subseteq X$, then $M \in \mathcal{N}(x)$.
- If $N \in \mathcal{N}(x)$ then there exists an $M \in \mathcal{N}(x)$ such that $N \in \mathcal{N}(y)$ for all $y \in M$.

We say that the function $\mathcal{N}$ defines a *topology* on X. Intuitively, the properties of $\mathcal{N}$ capture that a neighbourhood of x contains all points that are sufficiently close to x. It turns out that such a notion of closeness is all we need in order to define many important concepts. A central example is the continuity of functions.

Definition 2.2. A function $f\colon X \to Y$ is called *continuous* if for each $x \in X$ and each neighbourhood M of $f(x)$ there exists a neighbourhood N of x such that $f(N) \subseteq M$.

Another notion that we will often need is that of dense subsets.

Definition 2.3. A set $Y \subseteq X$ is *dense in* X if for every $x \in X$ and $N \in \mathcal{N}(x)$ we have $N \cap Y \neq \varnothing$.

If X admits linear combinations, for example if X is a vector space, we call Y *linearly dense in* X if the linear span of Y is dense in X. Finally, an important concept is that of an *open set*. We say that a set $U \subseteq X$ is *open* if it is a neighbourhood of all of its points, i.e. $U \in \mathcal{N}(x)$ for all $x \in U$. Intuitively, such a set has no "hard boundary": for each point in U, we can move a little bit in any direction and stay in the set. We call a set *closed* if its complement in X is open.

It turns out that there is an alternative, oftentimes more convenient formalism for defining topologies, where we instead start with the open sets of X.

Definition 2.4. A *topology* on X is a set τ of subsets of X, called open sets, satifying the following properties.

- $\varnothing \in \tau$ and $X \in \tau$
- τ is closed under unions: if $\{U_i \mid i \in I\} \subseteq \tau$ then $\bigcup_{i\in I} U_i \in \tau$.
- τ is closed under finite intersections: if $U_1, \dots, U_n \in \tau$ then $\bigcap_{i\in[n]} U_i \in \tau$.

Remark 2.5. It is easy to see that we can always define a topology by setting $\tau = 2^X$. This is called the *discrete topology*.

We can now define the neighbourhoods of a point $x \in X$ as the sets containing an open set containing x. One can then show that the two definitions of open sets, and those of neighbourhoods coincide. We can thus state a corresponding definition of continuous functions through open sets.

Definition 2.6. A function $f\colon X \to Y$ is called *continuous* if for every open set $U \subseteq Y$, the preimage $f^{-1}(U)$ is open.

We have seen that the entire topology of a set can be captured by defining which subsets are open. Consequently, when we reason about topological spaces, the question is usually "what happens on open sets?". This allows to define a powerful topological generalisation of finiteness: Suppose we have an infinite set $C \subseteq \tau$ of open sets, that together contain all points of X. Now if there always already exists a finite subset of C whose elements together contain all points of X, then in a sense there are only finitely many possibilities of "what happens on open sets". This property is called *compactness*.

Definition 2.7. A *cover* of a set X is a set $C = \{U_i \subseteq X \mid i \in I\}$ such that $\bigcup_{i\in I} U_i \supseteq X$. A *subcover* of C is a set $D \subseteq C$ that is also a cover of X. If X is a topological space with topology τ, we say that C is an *open cover* if $C \subseteq \tau$.

Definition 2.8. A topological space X is called *compact* if every open cover of X has a finite subcover.

There are many other properties along which we can classify topological spaces. However, we will only need one other concept, which relates to our ability to distinguish different points.

Definition 2.9. A topological space X is called *Hausdorff* if for every distinct $x, y \in X$ there exists a $U \in \mathcal{N}(x)$ and a $V \in \mathcal{N}(y)$ such that $U \cap V = \varnothing$.

We can now state a version of Urysohn's lemma, which we will later use to seperate arbitrary points of compact Hausdorff spaces by continuous functions.

Lemma 2.10 (Urysohn)**.** *Let X be a compact Hausdorff space and $x, y \in X$ be distinct points. Then there exists a continuous function $f\colon X \to \mathbb{C}$ such that $f(x) = 0$ and $f(y) = 1$.*

We conclude our treatment of purely topological spaces by defining a notion of isomorphism of topological spaces.

Definition 2.11. Let X, Y be topological spaces. A *homeomorphism* between X and Y is a function $f\colon X \to Y$ that is bijective, continuous, and an *open mapping*, i.e. maps open sets to open sets. A *self-homeomorphism* is a homeomorphism from a topological space to itself.

2.2.1 Metric Spaces

Of course, sometimes the topology on X is induced by stronger definitions. One example are metric spaces, where the notion of closeness is given by a concrete distance function $d\colon X \times X \to \mathbb{R}$. The open sets of a metric space X are those $U \subseteq X$ for which there exists an $\varepsilon \in \mathbb{R}$, such that for all $x \in U$ and $y \in X$ with $d(x, y) < \varepsilon$ we have $y \in U$. We can further define *bounded* sets, which are sets $U \subseteq X$ for which there exists an $r \in \mathbb{R}$ such that $d(x, y) \leq r$ for all $x, y \in U$. This allows us the state the following important characterisation of compactness.

Theorem 2.12 (Heine-Borel)**.** *Let S be a subset of $\mathbb{R}^n$. Then S is compact if and only if S is closed and bounded.*

A final question we can ask about a topological space is whether it has any "holes", that is, given a sequence $(a_n)_{n\in\mathbb{N}}$ that gets arbitrarily close to some point a, is $a \in X$? Formally, we say that $(a_n)_{n\in\mathbb{N}}$ *converges* to a if for every open neighbourhood U of a, there exists an $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$. For metric spaces, we can formalise this through *Cauchy sequences*.

Definition 2.13. A *Cauchy sequence* in X is a sequence $(a_n)_{n\in\mathbb{N}} \subset X$, such that for every $\varepsilon \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that $d(a_{n+1}, a_n) \leq \varepsilon$ for all $n > N$.

Definition 2.14. A metric space X is *complete* if every Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in X converges to an $a \in X$.

2.3 Notes on Category Theory

Since many results in the study of quantum groups have their origin in algebraic topology, some of the terminology is influenced by category theory. To better understand some of the later definitions, we want to introduce and motivate the central concepts.

A general observation is that certain results and definitions across different areas of mathematics look very similar, even though they might be phrased in very different formalisms. Through category theory, we seek to abstract away any formalism-inherent details and reduce mathematical concepts to the fundamental structure that is relevant in a given context. This allows us to prove very strong theorems that have applications across large swathes of mathematics. A striking example is Lawvere's fixed-point theorem [17], which implies well-known results like Cantor's theorem, Gödel's first incompleteness theorem, and the undecidability of the Halting Problem.

As the name "category theory" suggests, the central objects of interest are *categories*.

Definition 2.15. A *category* $\mathcal{C}$ consists of a collection $\mathrm{Obj}_{\mathcal{C}}$ of *objects*, denoted $X, Y, Z, \ldots$, and a collection of *morphisms* $\mathrm{Mor}_{\mathcal{C}}$, denoted $f, g, h, \ldots$, such that the following holds.

- Every morphism has a source and a target in $\mathrm{Obj}_{\mathcal{C}}$. We denote the class of morphisms with source X and target Y by $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ and also write $f\colon X \to Y$.
- For every object X there exists an *identity morphism* $\mathrm{id}_X\colon X \to X$.

- For every two morphisms $f\colon X \to Y$, $g\colon Y \to Z$, their *composition* $g \circ f$ is in $\mathrm{Mor}_{\mathcal{C}}(X, Z)$.

Moreover, it needs to satisfy the following axioms.

- *Unitality*: For every morphism $f\colon X \to Y$, the compositions $f \circ \mathrm{id}_X$ and $\mathrm{id}_Y \circ f$ are equal to f.
- *Associativity*: For morphisms $f\colon X \to Y$, $g\colon Y \to Z$, $h\colon Z \to W$, the morphisms $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are equal.

To specify a category, it thus suffices to define the objects, the morphisms, and how morphisms compose.

Example 2.16. A simple example is the category **Set**, given by

Objects:	Sets.
Morphisms:	Functions.
Composition:	Natural composition.

⌟

Example 2.17. The vector spaces over a field $\mathbb{K}$ form a category, called $\mathbf{Vect}_{\mathbb{K}}$, as follows.

Objects:	$\mathbb{K}$-vector spaces.
Morphisms:	Vector space homomorphisms / linear maps.
Composition:	Natural composition.

⌟

Example 2.18. The morphisms generally do not have to be functions. Consider the category $\mathbf{Set}^{\mathrm{op}}$ obtained from **Set** by reversing all morphisms, i.e. swapping source and target. Then there exists a morphism $f \in \mathrm{Mor}_{\mathbf{Set}^{\mathrm{op}}}(X, \varnothing)$ for all $X \in \mathrm{Obj}_{\mathbf{Set}^{\mathrm{op}}}$, even though no such function exists. ⌟

We call a morphism $f\colon X \to Y$ an *isomorphism* if there exists a $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In that case we call X and Y *isomorphic*. The morphisms in **Set** are thus simply the bijective functions; in **Vect**, the notion of isomorphism coincides with the non category-theoretic definition.

A very powerful tool to reason about categories are *diagrams*. These are graphical representations of categories, where objects are visualized as points, and morphisms as arrows between them. We usually omit the identity morphisms and the compositions of morphisms, which are always implicit. We say that a diagram *commutes* if for each pair of objects X, Y in the diagram, composing along arbitrary paths between X and Y always yields the same morphism.

Example 2.19. Consider the following diagram of objects in **Set** and a binary function f.

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{f \times \mathrm{id}_X} & X \times X \\ \downarrow{\scriptstyle \mathrm{id}_X \times f} & & \downarrow{\scriptstyle f} \\ X \times X & \xrightarrow{\quad f \quad} & X \end{array}$$

It commutes if and only if $(f \times \mathrm{id}_X) \circ f = (\mathrm{id}_X \times f) \circ f$, that is, the function f is associative. ⌟

A useful technique is *dualisation*, where we reverse all morphisms to obtain an *opposite* category, like we did in Example 2.18 for **Set**. In terms of diagrams, this is corresponds to "flipping the arrows". Since the definition of categories is symmetric with respect to this reversal, every statement we prove about categories also gives us a *dual* statement that holds in the opposite category. We often indicate these dual properties by the prefix "co-".

Example 2.20. Consider Example 2.19. If f is associative and thus the above diagram commutes, then the diagram

$$\begin{array}{ccc} X \times X \times X & \xleftarrow{f^{\mathrm{op}} \times \mathrm{id}_X} & X \times X \\ {\scriptstyle \mathrm{id}_X \times f^{\mathrm{op}}}\uparrow & & \uparrow{\scriptstyle f^{\mathrm{op}}} \\ X \times X & \xleftarrow[f^{\mathrm{op}}]{} & X \end{array}$$

commutes in $\mathbf{Set}^{\mathrm{op}}$ and we call the morphism f^{op} corresponding to f *coassociative*. ⌟

Sometimes we are interested in the relation between different categories. To that end, we need a notion of mappings between categories.

Definition 2.21. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ maps

- each object $X \in \mathrm{Obj}_{\mathcal{C}}$ to an object $FX \in \mathrm{Obj}_{\mathcal{D}}$,
- each morphism $f\colon X \to Y$ to a morphism $Ff\colon FX \to FY$,

such that the following axioms are satisfied.

- *Unitality*: For all $X \in \mathrm{Obj}_{\mathcal{C}}$, $F(\mathrm{id}_X) = \mathrm{id}_{FX}$.
- *Compositionality*: For every $f\colon X \to Y$, $g\colon Z \to Z$, we have $F(g \circ f) = F(g) \circ F(f)$.

Example 2.22. The *forgetful functor* F_* maps a category of structures to **Set**, forgetting the relations and functions. In the case of $\mathbf{Vect}_{\mathbb{K}}$, it is given by

On objects:	$(V, +, (\cdot k)_{k \in \mathbb{K}}) \mapsto V$.
On morphisms:	$f \mapsto f$ for all linear maps f.

It easy to see that F_* satisfies the functoriality axioms. ⌟

We call a functor $F\colon \mathcal{C} \to \mathcal{D}$ *full*, if the map $\mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathcal{D}}(FX, FY)$ is surjective for all $X, Y \in \mathrm{Obj}_{\mathcal{C}}$. We call it *faithful* if this map is injective. F is called *fully faithful* if it is full and faithful. Finally, we call F *essentially surjective* if for every object $D \in \mathrm{Obj}_{\mathcal{D}}$ there exists an object $C \in \mathrm{Obj}_{\mathcal{C}}$ such that D and FC are isomorphic.

It turns out that many mathematical concepts are best represented by functors instead of categories themselves. Hence, we would also like to study the relation between functors. To that end we use natural transformations.

Definition 2.23. Let $\mathcal{C}, \mathcal{D}$ be categories and $E, F\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\alpha\colon E \Rightarrow F$ from E to F is a collection of morphisms $\alpha_X \in \mathrm{Mor}_{\mathcal{D}}(EX, FX)$ for $X \in \mathrm{Obj}_{\mathcal{C}}$, such that for each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, we have $Ff \circ \alpha_X = \alpha_Y \circ Ef$, i.e. the following diagram commutes

$$\begin{array}{ccc} EX & \xrightarrow{Ef} & EY \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ FX & \xrightarrow{Ff} & FY \end{array}$$

If all the α_X are isomorphisms, we call α a *natural isomorphism*.

3 Quantum Groups

> "But I don't want to go among mad people," Alice remarked.
> "Oh, you can't help that," said the Cat: "we're all mad here. I'm mad. You're mad."
> "How do you know I'm mad?" said Alice.
> "You must be," said the Cat, "or you wouldn't have come here."
>
> *Lewis Carroll, Alice in Wonderland*

Groups describe *symmetries*; and symmetries are omnipresent. Almost all physical laws of nature seem to originate in symmetries—and as Anderson writes in [1], "it is only slightly overstating the case that physics is the study of symmetry". Unsurprisingly, groups are effective mathematical tools in physical theories. *Lie groups*, in particular, are a natural choice to describe continuous symmetries, and are regularly encountered in elementary particle physics.

However, it is a fact of life that a physical theory generally describes only a limited class of phenomena, and applying a theory beyond this range of validity yields results increasingly disagreeing with experimental reality. In order to describe these new phenomena, one has to modify the theory. One way to do this is through a process called *deformation*.

Suppose a Lie group G is part of a physical theory. A deformation of G is a family of groups $(G_\tau)_{\tau\in[0,\varepsilon)}$ such that $G_0 = G$ and G_τ depends continuously on τ in some way. The new theory is then obtained by replacing G by G_{τ_0} for some $\tau_0 \in (0,\varepsilon)$. The value τ_0 becomes one of the fundamental constants of the new theory, and we recover the old theory in the limit $\tau_0 \to 0$.

At first glance, it seems surprising that such an approach works to extend the range of validity of a theory—but indeed there are many examples in the history of physics that fit into this scheme. Woronowicz mentions some of these theories in [33], a particularly notable example being quantum mechanics.[1]

However, there are areas where this approach reaches its limits. One example is elementary particle physics, where many results are phrased in terms of so called compact semisimple Lie groups. Unfortunately, these groups only admit trivial deformations, where all G_τ are isomorphic to G. To remedy this, one might relax the requirements of these groups by moving to more abstract topological spaces that still contain the essence of the original group structure. These *quantum groups* then admit a much larger class of deformations and allow defining non-trivial deformations where they did not exist before.

Historically, this was the motivation for studying compact quantum groups. Since then, they have proved an interesting object of study in their own right. Apart from physics, they have applications in free probability [30], which deals with the probability theory of non-commutative random variables. We will see that in a certain sense quantum groups describe *non-commutative* symmetries.

The way we obtain these quantum groups is through a well-known duality between topological spaces and the algebra of complex-valued functions on them. This function algebra is always commutative, and for compact Hausdorff spaces we can recover the original space from the algebra. To some extent, this means that we can reason about the topological space through its function algebra. We now define quantum groups by instead *starting* with some algebra A and asking which topological space induces it as its function algebra. Obviously, if A is non-commutative no such space exists, but we can view the quantum group as the *virtual object* that has A as its function algebra. As long as we only reason about the algebra, it does not matter

[1] In quantum mechanics, the τ_0 in question is Planck's constant $\hbar$, and we recover classical mechanics as $\hbar \to 0$, "in the classical limit".

whether the underlying object actually exists. In this sense, quantum groups generalise classical groups; and the classical groups are precisely those quantum groups whose function algebra is commutative.

In this chapter, we want to demystify this approach. We will start at the very beginning and introduce the fundamental ideas underlying the theory of quantum groups. We will then be able to formally define compact quantum groups and compact matrix quantum groups. After the general definitions, we will investigate how certain group-theoretic concepts can be generalised to the quantum case. Specifically, we will look at group actions and (quantum) orbits. Finally, we will turn our attention to easy quantum groups. These compact matrix quantum groups enjoy a purely combinatorical characterisation and have been fully classified by Raum and Weber [26]. They form the starting point of our investigation in the following chapters.

3.1 C*-algebras

Central to our definition of quantum groups are *algebras*, which are vector spaces that allow for multiplication of vectors.

Definition 3.1. An algebra A is a vector space equipped with a bilinear multiplication map $A \times A \to A$. We will always assume A to be *associative* and *unital*, that is, the multiplication is associative and there exists a multiplicative unit $\mathbf{1} \in A$.

Example 3.2. The rational numbers $\mathbb{Q}$ form an algebra (over $\mathbb{Q}$) together with the standard addition and multiplication. Here, the bilinearity condition is equivalent to the distributivity of multiplication. ⌟

We are often interested in algebras with additional structure, such as algebras over metric spaces. One example of metric spaces are normed vector spaces: Suppose V is a vector space that admits a norm $\|\cdot\| \colon V \to \mathbb{R}$. We can define a distance on V by $d(x, y) = \|x - y\|$, which turns it into a metric space. From there, we can ask whether the space is complete, that is, whether for every Cauchy sequence $x_1, x_2, \dots$ there exists an $x \in V$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

Definition 3.3. A *Banach space* is a complete normed vector space.

Definition 3.4. A *Banach algebra* B is an algebra that is also a Banach space whose norm is submultiplicative, i.e. that satisfies $\|xy\| \leq \|x\| \cdot \|y\|$ for all $x, y \in B$.

Example 3.5. The real numbers $\mathbb{R}$ form a Banach algebra (over $\mathbb{R}$). It is an algebra for the same reasons as in Example 3.2. Moreover, the absolute value $|x|$ defines a norm that is complete per definition of the reals. ⌟

C*-algebras were originally introduced to describe bounded linear operators on a Hilbert space (a Banach space over $\mathbb{R}$ or $\mathbb{C}$ whose norm is induced by an inner product). These operators admit an involutive mapping called the *hermitian adjoint*, that we want the algebra to capture.

Definition 3.6. A **-algebra* A is an algebra over the complex numbers equipped with an involutive antiautomorphism $^* \colon A \to A$ such that

$$(\lambda x)^* = \overline{\lambda} x^*$$

for all $\lambda \in \mathbb{C}$ and $x \in A$.

We also want a norm that is compatible with the adjoint mapping.

Definition 3.7. A *C*-norm* $\|\cdot\| \colon A \to \mathbb{R}$ on a *-algebra A is a norm that satisfies

$$\|x^* x\| = \|x\|^2 \quad \text{and} \quad \|xy\| \leq \|x\| \cdot \|y\|$$

for all $x, y \in A$.

Definition 3.8. A *C*-algebra* is a *-algebra over $\mathbb{C}$ that is also a Banach algebra with respect to a C*-norm.

Example 3.9. The complex numbers $\mathbb{C}$ form a C*-algebra. They form a *-algebra with the standard addition and multiplication, and complex conjugation as an involutive antiautomorphism. Moreover, the norm of complex numbers is defined as $\|x\| = \sqrt{x^*x}$, so we get $\|x^*x\| = \sqrt{(x^*x)^*x^*x} = \sqrt{(x^*x)^2} = x^*x = \|x\|^2$, and $\|xy\| = \sqrt{(xy)^*xy} = \sqrt{x^*xy^*y} = \sqrt{x^*x}\sqrt{y^*y} = \|x\| \cdot \|y\|$. Finally, the completeness of $\mathbb{C}$ follows from the completeness of $\mathbb{R}$, since $\mathbb{C} \cong \mathbb{R}^2$ as metric spaces. ⌟

Example 3.10. Let H be a Hilbert space, and let

$$\mathcal{L}(H) = \{T\colon H \to H \mid T \text{ linear and bounded}\}$$

be the set of bounded linear operators on H. Then $\mathcal{L}(H)$ forms a C*-algebra. Let $\langle \cdot, \cdot \rangle$ be the inner product of H. The *hermitian adjoint* of an operator T is defined as the unique operator T^* such that $\langle x, Ty \rangle = \langle T^*x, y \rangle$ for all $x, y \in H$. Together with pointwise addition and composition as bilinear multiplication, this turns $\mathcal{L}(H)$ into a *-algebra. Finally, one can show that the operator norm

$$\|T\| = \inf\{C \geq 0 \mid \|Tx\| \leq C\|x\| \text{ for all } x \in H\}$$

turns it into a C*-algebra. In fact, it turns out that C*-algebras actually capture all of the structure of bounded linear operators: By the Gelfand-Naimark theorem, every C*-algebra is isomorphic to a norm-closed subalgebra of $\mathcal{L}(H)$; see [4, Corollary II.6.4.10]. ⌟

The interpretation of C*-algebras as operator algebras motivates the following definitions.

Definition 3.11. Let A be a *-algebra. An element $x \in A$ is called

- *self-adjoint* if $x^* = x$,
- *unitary* if $x^*x = xx^* = \mathbf{1}$,
- *projection* if $x^2 = x^* = x$,
- *positive* if $x = y^*y$ for some $y \in A$.

Finally, one may ask what the correct notion of a homomorphism between C*-algebras should be. For two algebras A, B, we can define an algebra homomorphism $h\colon A \to B$ as a linear map satisfying $h(xy) = h(x)h(y)$ for $x, y \in A$. If A and B are *-algebras, we call h a **-homomorphism* if additionally $h(x^*) = h(x)^*$. One can show that any *-homomorphism between C*-algebras must be continuous, which makes them the natural choice of homomorphism for C*-algebras as well.

3.1.1 Tensor Products

Recall the tensor product $A \otimes B$ of vector spaces A and B. We can extend this definition to algebras A, B by defining the multiplication on $A \otimes B$ as

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2.$$

If furthermore A and B are *-algebras, we can define a *-algebra structure on $A \otimes B$ by setting $(a \otimes b)^* = a^* \otimes b^*$, but it is generally not clear how to define the tensor product of two C*-algebras. Concretely, the *-algebra tensor product $A \otimes B$ is generally not complete with respect to a C*-norm.

One way to define the tensor product regardless is to consider the completion of $A \otimes B$ with respect to an arbitrary C*-norm. This however means that the C*-structure on $A \otimes B$ is not uniquely determined by A and B. An attempt at a canonical choice is through so called *universal C*-enveloping algebras*, which are useful in other contexts as well.

Definition 3.12. Let A be a *-algebra and set

$$\|a\| := \sup\{\rho(a) \mid \rho \text{ is a C}^*\text{-norm on } A\}$$

for all $a \in A$. If $\|a\| < \infty$ for all $a \in A$, we define the *universal C*-enveloping algebra* $\hat{A}$ of A as the completion of

$$A \Big/ \{a \in A \mid \|a\| = 0\}.$$

One can verify that the function $\|\cdot\|$ satisfies the C*-norm properties, the triangle inequality, and $\|\lambda x\| = |\lambda| \cdot \|x\|$ for $x \in A$ and $\lambda \in \mathbb{C}$. Consequently, quotienting out elements x with $\|x\| = 0$ turns it into a C*-norm. Finally, adding the limit points of Cauchy sequences turns $\hat{A}$ into a C*-algebra.

For $A \otimes B$ specifically, one can show that $\rho(a \otimes b) \leq \|a\|_A \cdot \|b\|_B$ for all C*-norms ρ on $A \otimes B$; see e.g. [4, Corollary II.9.2.2]. This means that the universal C*-enveloping algebra of $A \otimes B$ exists, which is called the *maximal* tensor product of A and B and is denoted by $A \otimes_{max} B$. Conversely, through the correspondence between C*-algebras and bounded linear operators one can also define a *minimal* tensor product $A \otimes_{min} B$. These tensor products are minimal and maximal in the following sense. Let $\|\cdot\|_{min}$ and $\|\cdot\|_{max}$ be the C*-norms associated to $A \otimes_{min} B$ and $A \otimes_{max} B$. Then $\|x\|_{min} \leq \rho(x) \leq \|x\|_{max}$ for all C*-norms ρ and $x \in A \otimes B$; see [4, pp. II.9.1.3, II.9.5.2].

Remark 3.13. If $A \otimes B$ is finite-dimensional, it is already complete with respect to any norm and thus already forms a C*-algebra. Moreover, one can show that the norm on a C*-algebra is unique, so we have $A \otimes B = A \otimes_{max} B = A \otimes_{min} B$.

For simplicity, we will no longer distinguish between different tensor products in the following chapters. Whenever A and B are C*-algebras, we understand $A \otimes B$ as the completion of the *-algebra tensor product of A and B with respect to some C*-norm.

3.1.2 The Algebra of Complex Valued Functions

To define quantum groups, we are mostly interested in one particular family of C*-algebras. Let X be a compact Hausdorff space. Then the *continuous complex-valued functions*

$$C(X) = \{\phi \colon X \to \mathbb{C} \mid \phi \text{ continuous}\}$$

form a commutative *-algebra under pointwise addition, multiplication and complex conjugation. We define the *supremum norm* as

$$\|\phi\| = \sup|\operatorname{img}\phi| = \sup\{|\phi(x)| \mid x \in X\}.$$

Note that the norm is well-defined: Since X is compact and ϕ is continuous, $\operatorname{img}\phi \subseteq \mathbb{C}$ is also compact and thus, by the Heine-Borel theorem, bounded. A straightforward calculation moreover shows that the supremum norm is a C*-norm. With a bit more work we can further show that $C(X)$ is complete under this norm, and it follows that it is indeed a C*-algebra.

The well-known Gelfand-Naimark theorem now says that really any commutative C*-algebra is of this form.

Theorem 3.14 (Gelfand-Naimark)**.** *Let A be a commutative C*-algebra. Then there exists a compact Hausdorff space such that $C(X) \cong A$.*

This means that we can identify commutative C*-algebras with function algebras over compact Hausdorff spaces. To define quantum groups we will later go a step further and identify *non-commutative* C*-algebras with function algebras over virtual "non-commutative topological spaces".

Before we turn our attention to groups, we want to highlight a convenient property of commutative C*-algebras that allows us to identify the tensor product of function algebras with another function algebra.

Lemma 3.15. *Let X_1 and X_2 be compact Hausdorff spaces. Then the mapping*

$$C(X_1) \otimes C(X_2) \to C(X_1 \times X_2), f \otimes g \mapsto (X_1 \times X_2 \to \mathbb{C}, (x,y) \mapsto f(x)g(y))$$

is an isomorphism of C-algebras.*

For simplicity, we will often tacitly identify $C(X_1) \otimes C(X_2)$ and $C(X_1 \times X_2)$.

3.2 Compact Quantum Groups

Having established the mathematical foundations, we now turn to defining quantum groups. As noted above, we will make use of the duality between certain topological spaces and function algebras. Concretely, this means that we want to work with groups that are also topological spaces.

Definition 3.16. A *topological group* is a group $(X, \cdot)$ whose underlying set X is endowed with a topology such that the group operation $\cdot : X \times X \to X$, and the inversion map $^{-1}: X \to X$ are continuous.

Definition 3.17. A *compact group* is a topological group whose underlying space is a compact Hausdorff space.

Example 3.18. Any group is a topological group with respect to the discrete topology. It is compact if and only if it is finite. ⌟

Example 3.19. The circle group $\mathbb{S}^1$, that is, the multiplicative group of the complex numbers of absolute value 1, is a compact group. The inversion map is given by the complex conjugate, and the group is compact as a closed and bounded subset of $\mathbb{C}$. ⌟

3.2.1 Generalising Compact Groups

We have seen that there is a correspondence between compact Hausdorff spaces and commutative C*-algebras. Namely, if X is a compact Hausdorff space then the space of continuous complex-valued functions $C(X)$ forms a commutative C*-algebra; and conversely for every commutative C*-algebra A there exists a compact Hausdorff space X, such that $A \cong C(X)$. In this regard, we may view non-commutative C*-algebras as function algebras $C(X^+)$ of some "non-commutative" topological space X^+. Of course, no such space actually exists, as complex-valued functions are always commutative under pointwise multiplication. We simply treat X^+ as a virtual object associated to the algebra.

Now let $(G, \cdot)$ be a compact group. Then G is in particular a compact Hausdorff space. This suggests to define quantum groups exactly through the above correspondence: as underlying topological groups of (potentially non-commutative) C*-algebras. There is, however, an obstacle to this. $C(G)$ forms a commutative C*-algebra; but given a commutative C*-algebra A, we can only find a general topological space X with $A \cong C(X)$, not necessarily a topological group. In other words, we forget the group structure when we move to the function algebra.

Towards a definition of quantum groups, observe that the multiplication of G induces a *comultiplication* on $C(G)$, which is a *-homomorphism given by

$$\Delta: C(G) \to C(G) \otimes C(G), \phi \mapsto [G \times G \to \mathbb{C}, (x,y) \mapsto \phi(xy)]. \tag{3.1}$$

This can be seen as the *dual* of the multiplication in the category-theoretic sense, so the associativity of G implies that Δ is *coassociative*, that is, $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$, or the following diagram commutes.

$$\begin{array}{ccc} C(G) & \xrightarrow{\Delta} & C(G)\otimes C(G) \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ C(G)\otimes C(G) & \xrightarrow[\Delta\otimes\mathrm{id}]{} & C(G)\otimes C(G)\otimes C(G) \end{array}$$

This alone is enough to capture the associativity of G, which motivates the following definition.

Definition 3.20. A *compact quantum semigroup* G is given by a tuple $(C(G), \Delta)$ consisting of a C*-algebra $C(G)$ and a *-homomorphism $\Delta\colon C(G) \to C(G)\otimes C(G)$, called comultiplication, such that $(\mathrm{id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{id})\circ\Delta$.

We can recover a semigroup from any commutative algebra A with a coassociative comultiplication Δ. By Theorem 3.14, A is isomorphic to $C(X)$ for some compact Hausdorff space X. Moreover, it follows from Urysohn's lemma that the points of X can be distinguished by functions $f \in C(X)$. Hence, $(\Delta(f))(x,y) =: f(xy)$ uniquely defines an associative multiplication on X.

Lemma 3.21. *Let A be a commutative C*-algebra together with a comultiplication satisfying the requirements of Definition 3.20. Then there exists a compact semigroup G and an isomorphism $A \to C(G)$ that preserves the comultiplication. Moreover, G is unique up to homeomorphism.*

To further capture that G is a group, we need a way to encode the inversion map and the existence of a neutral element e. In the spirit of Equation (3.1), we might define the maps

$$\varepsilon\colon C(G) \to \mathbb{C}, \phi \mapsto \phi(e),$$

$$\kappa\colon C(G) \to C(G), \phi \mapsto [G \to \mathbb{C}, a \mapsto \phi(a^{-1})].$$

These maps give us a *counit* and a *coinverse* that capture the relevant properties that make G a group. Moreover, together with the comultiplication they turn $C(G)$ into a *Hopf algebra*. These algebras are well-studied and have many desirable properties, so it is tempting to define a compact quantum group as the virtual object G such that the C*-algebra $C(G)$ of continuous complex-valued functions on G is also a Hopf algebra.

Unfortunately, it turns out that ε and κ are generally very badly behaved with respect to the C*-structure when $C(G)$ is non-commutative; see [21]. An alternative approach, taken by Woronowicz in [35], is based on the following group-theoretic observation.

Lemma 3.22. *A compact semigroup G is a group if and only if it satisfies the cancellation property, that is, $xz = yz$ and $zx = zy$ each imply $x = y$ for all $x, y, z \in G$.*

We can translate the cancellation property in terms of the comultiplication Δ.

Lemma 3.23. *Let G be a compact semigroup, and let Δ be defined as in Equation (3.1). Then G has the cancellation property if the sets $\Delta(C(G))(\mathbf{1}\otimes C(G))$ and $\Delta(C(G))(C(G)\otimes\mathbf{1})$ are linearly dense in $C(G)\otimes C(G)$.*

Proof. Suppose first that $\Delta(C(G))(\mathbf{1}\otimes C(G))$ is dense in $C(G)\otimes C(G)$. Let $x, y, z \in G$ such that $xz = yz$. Then we have

$$(\Delta(f)(\mathbf{1}\otimes g))(x,z) = f(xz)g(z) = f(yz)g(z) = (\Delta(f)(\mathbf{1}\otimes g))(y,z)$$

for all $f, g \in C(G)$. Since $\Delta(C(G))(\mathbf{1}\otimes C(G))$ is linearly dense in $C(G)\otimes C(G)$, we have $h(x,z) = h(y,z)$ for all $h \in C(G)\otimes C(G)$ and thus $x = y$. This implies that G is right-cancellative. A symmetric argument shows that $\Delta(C(G))(C(G)\otimes\mathbf{1})$ being linearly dense in $C(G)\otimes C(G)$ implies that G is left-cancellative. □

Conversely, we can show that for compact groups our choice of comultiplication satisfies this property.

Lemma 3.24. *Let G be a compact group, and let Δ be defined as in Equation* (3.1)*. Then the sets $\Delta(C(G))(\mathbf{1} \otimes C(G))$ and $\Delta(C(G))(C(G) \otimes \mathbf{1})$ are linearly dense in $C(G) \otimes C(G)$.*

Proof (*Sketch*). Since G is a compact group, one can show that the map

$$h\colon (x, y) \mapsto (xy, y)$$

is a self-homeomorphism of $G \times G$. It induces an automorphism of $C(G) \otimes C(G)$ as follows. Recall that we can identify $f \otimes g \in C(G) \otimes C(G)$ with $(x, y) \mapsto f(x)g(y)$. Composed with h, this function uniquely corresponds to the map $(x, y) \mapsto f(xy)g(y)$, which is just equal to $\Delta(f)(\mathbf{1} \otimes g)$. Hence

$$f \otimes g \mapsto \Delta(f)(\mathbf{1} \otimes g)$$

is our desired automorphism. Similarly, one can show that the map $(x, y) \mapsto (x, xy)$ is a homeomorphism, which in turn implies that $f \otimes g \mapsto \Delta(f)(g \otimes \mathbf{1})$ is an automorphism. In particular, it follows that $\Delta(C(G))(\mathbf{1} \otimes C(G))$ and $\Delta(C(G))(C(G) \otimes \mathbf{1})$ are dense in $C(G) \otimes C(G)$. □

Finally, we can give a definition of compact quantum groups.

Definition 3.25. A *compact quantum group* G is given by a tuple $(C(G), \Delta)$ consisting of a C*-algebra $C(G)$ and a *-homomorphism $\Delta\colon C(G) \to C(G) \otimes C(G)$, called comultiplication, such that

(i) $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$.

(ii) $\Delta(C(G))(\mathbf{1} \otimes C(G))$ and $\Delta(C(G))(C(G) \otimes \mathbf{1})$ are linearly dense in $C(G) \otimes C(G)$.

In analogy to Lemma 3.21, Lemmas 3.22 and 3.23 imply that we can recover a unique compact group whenever the C*-algebra is commutative.

Lemma 3.26. *Let A be a commutative C*-algebra together with a comultiplication satisfying the requirements of Definition 3.25. Then there exists a compact group G and an isomorphism $A \to C(G)$ that preserves the comultiplication. Moreover, G is unique up to homeomorphism.*

3.2.2 Compact Matrix Quantum Groups

One subclass of compact quantum groups is of particular interest to us: the compact matrix quantum groups, which generalise compact matrix groups.

Definition 3.27. A compact group G is called a *compact matrix group* if it admits a *faithful matrix representation*, that is, an injective continuous group homomorphism $\chi\colon G \to \mathrm{GL}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

Remark 3.28. Due to the injectivity of χ, we can identify G and $\chi(G)$. Moreover, one can show that any compact subgroup of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a subgroup of U_n, so we may further assume that $G \subseteq U_n$.

Again, the question is how one can mirror this property in $C(G)$. The crucial observation is that in a certain sense $\chi(G)$ generates $C(G)$. To see this, we define the coordinate functions

$$u_{ij}\colon G \to \mathbb{C}, x \mapsto \chi(x)_{ij}$$

for $i, j \in [n]$, which map a group element to the ij-th entry of its matrix representation. These functions are continuous, and obviously seperate the elements of G in the sense that for $x \neq y \in G$,

there exist $i, j \in [n]$ with $u_{ij}(x) \neq u_{ij}(y)$. This means that by the Stone-Weierstraß theorem, the *-algebra generated by $\{u_{ij} \mid i, j \in [n]\}$ is dense in $C(G)$.

Moreover, given this observation the comultiplication on $C(G)$ takes a convenient form. Recall the comultiplication defined in Equation (3.1), which turns the compact group G into a compact quantum group. For the generators u_{ij}, the definition simplifies to

$$\Delta(u_{ij})\colon G \times G \to \mathbb{C}, (x, y) \mapsto u_{ij}(xy) = \sum_{k=1}^{n} x_{ik} y_{kj} = \sum_{k=1}^{n} u_{ik}(x) u_{kj}(y),$$

which, under the isomorphism given in Lemma 3.15, corresponds to $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$. The definition of the generators suggests to arrange them in a matrix $\mathcal{U} = (u_{ij}) \in C(G)^{n \times n}$, and since $G \subseteq U_n$, we find that $\mathcal{U}$ is unitary in the sense that $\mathcal{U}^*\mathcal{U} = \mathcal{U}\mathcal{U}^* = \mathbf{1}I$, where $\mathcal{U}^* = (u_{ji}^*)$. A quick calculation also shows that its transpose $\mathcal{U}^\mathsf{T}$ is invertible with inverse $\overline{\mathcal{U}} = (u_{ij}^*)$.

It turns out that these properties are already strong enough to capture the (quantum) group structure of G. More precisely, one can prove the following.

Lemma 3.29 (Proposition 3.8 in [21])**.** *Let A be a C*-algebra generated by the elements of a matrix $\mathcal{U} = (u_{ij})$ such that $\mathcal{U}$ and its transpose $\mathcal{U}^\mathsf{T}$ are invertible, and*

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

is a *-homomorphism. Then (A, Δ) forms a compact quantum group.

This suggests the following definition of compact matrix quantum groups.

Definition 3.30. A *compact matrix quantum group* G is given by a tuple $(C(G), \mathcal{U}, n)$, consisting of a C*-algebra $C(G)$, an integer $n \in \mathbb{N}$ and a matrix $\mathcal{U} = (u_{ij}) \in C(G)^{n \times n}$, called the *fundamental representation* of G, such that

(i) The elements $\{u_{ij} \mid i, j \in [n]\}$ generate $C(G)$ in the sense that the generated *-algebra is dense in $C(G)$.

(ii) The matrix $\mathcal{U}$ is unitary and $\mathcal{U}^\mathsf{T}$ is invertible.

(iii) The map $\Delta\colon C(G) \to C(G) \otimes C(G), u_{ij} \mapsto \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ is a *-homomorphism, called comultiplication.

As for semigroups and groups, in the commutative case these requirements are sufficient to revocer a compact matrix group G. We have already seen with Lemma 3.26 that every compact quantum group with a commutative underlying C*-algebra A uniquely corresponds to a compact group G, that is, there exists an isomorphism $\phi\colon A \to C(G)$ that preserves the comultiplication. From there one can show with a bit of work that the mapping χ defined by $\chi(x)_{ij} = \phi(u_{ij})(x)$ is an injective continuous group homomorphism $\chi\colon G \to \mathrm{GL}_n(\mathbb{C})$, which turns G into a compact matrix group.

Lemma 3.31. *Let A be a commutative C*-algebra together with a matrix $\mathcal{U} \in A^{n \times n}$ satisfying the requirements of Definition 3.30. Then there exists a compact matrix group $G \subseteq U_n$ and an isomorphism $A \to C(G), u_{ij} \mapsto v_{ij}$ such that $(C(G), (v_{ij}), n)$ forms a compact matrix quantum group. Moreover, G is unique up to homeomorphism.*

3.2.3 Morphisms and Subgroups

Before we look at examples of compact matrix quantum groups, let us extend some fundamental group-theoretic notions to the quantum case.

Definition 3.32. A homomorphism between two compact quantum groups $G = (C(G), \Delta_G)$ and $H = (C(H), \Delta_H)$ is a *-homomorphism $h\colon C(G) \to C(H)$ such that

$$(h \otimes h) \circ \Delta_G = \Delta_H \otimes h.$$

For compact matrix quantum groups, we want homomorphisms to moreover respect the fundamental representations.

Definition 3.33. A homomorphism between two compact matrix quantum groups $G = (C(G), \mathcal{U}, n)$ and $H = (C(H), \mathcal{V}, n)$ is a *-homomorphism $h\colon C(G) \to C(H)$ such that

$$T\mathcal{V} = h(\mathcal{U})T$$

for some matrix $T \in \mathrm{GL}_n(\mathbb{C})$, where $h(\mathcal{U})$ denotes entrywise application of h. If h is an isomorphism, we call G and H *similar*.

One can show that every homomorphism between two compact matrix quantum groups is also a homomorphism of compact quantum groups in the sense of Definition 3.32.

With these definitions, we can define quantum subgroups.

Definition 3.34. Let G, H be compact quantum groups. Then H is a *compact quantum subgroup* of G if there exists a surjective homomorphism $h\colon G \to H$.

For compact matrix quantum groups we again go a step further and require the surjective homomorphism to preserve the fundamental representation.

Definition 3.35. Let $G = (C(G), \mathcal{U}, n)$ and $H = (C(H), \mathcal{V}, n)$ be compact matrix quantum groups. Then H is a *compact matrix quantum subgroup* of G if there exists a surjective homomorphism $h\colon G \to H$ such that $\mathcal{V} = h(\mathcal{U})$.

Remark 3.36. Let $G, H \subseteq U_n$ be two compact matrix groups. Then it is straightforward to verify that H is a compact matrix quantum subgroup of G if and only if H is a subgroup of G.

3.2.4 Important examples

We now want to highlight some salient examples of compact matrix quantum groups that we will also come back to later in the context of homomorphism indistinguishability. However, contrary to classical matrix groups, whose fundamental representations as matrix quantum groups can be given in terms of coordinate functions mapping the group elements to their matrix entries, non-classical groups—compact matrix quantum groups with non-commutative C*-algebras—can be more difficult to define. We must find a fundamental representation that somehow encodes the properties that we want the quantum group to have. Usually, we have a set of relations in mind that we want the elements of the C*-algebra to satisfy. One way to achieve this is through universal C*-algebras.

Definition 3.37. Let Σ be a set of symbols and let $\mathbb{K}$ be a field. Let $\Sigma^* = \{x^* \mid x \in \Sigma\}$. Then the *free *-algebra over* Σ, denoted $\mathbb{K}\langle\Sigma\rangle^*$, is the algebra over $\mathbb{K}$ given by the linear span of the words over the alphabet $\Sigma \cup \Sigma^*$, where multiplication is given by concatenation.

In other words, $\mathbb{K}\langle\Sigma\rangle^*$ is the *-algebra generated by the symbols Σ. Given the syntactic nature of the free algebra, it is easy to define the relations that the algebra elements should satisfy. Note that we can view relations of the form $\alpha = \beta$ as elements of $\mathbb{K}\langle\Sigma\rangle^*$. Indeed, if $x = y$ for $x, y \in \mathbb{K}\langle\Sigma\rangle^*$ then $x - y = 0$ and vice versa. This means that to satisfy the relations, we can simply "quotient out" these elements.

Definition 3.38. Let Σ be a set of symbols and $R \subseteq \mathbb{C}\langle\Sigma\rangle^*$ a set of relations. Consider the two-sided ideal $J(R)$ given by the *-closure of the ideal generated by R and let

$$\langle \Sigma \mid R\rangle := \mathbb{C}\langle\Sigma\rangle^* \Big/ J(R).$$

If the universal C*-enveloping algebra of $\langle \Sigma \mid R\rangle$ exists, we denote it by $C^*\langle \Sigma \mid R\rangle$ and call it the *universal C*-algebra with generators Σ and relations R*.

Note that by this definition, the universal C*-algebra only exist if the relations impose an upper bound on all C*-norms.

Example 3.39. Consider $A := \langle u \mid u^* = u, u^2 = \mathbf{1}\rangle$. Then a C*-norm $\|\cdot\|$ on A must satisfy $\|\mathbf{1}\| = 1$ and $\|u\|^2 = \|u^*u\| = \|u^2\| = \|\mathbf{1}\|$. Since any C*-norm is subadditive and submultiplicative, it follows that $\|x\|$ is bounded for all $x \in A$, and the universal C*-algebra $C^*\langle u \mid u^* = u, u^2 = \mathbf{1}\rangle$ exists. In fact, one can show that it is isomorphic to the continuous real-valued functions on the unit circle, cf. [5, Example 1.3].

On the other hand, the universal algebra with a single generator u and relation $u^* = u$ does not exist. The underlying algebra $B := \langle u \mid u^* = u\rangle$ is isomorphic to the ring $\mathbb{C}[x]$ of complex polynomials, but for any $k > 0$, we can find a C*-norm assigning $\|u\| = k$. Hence we have $\sup\{\rho(u) \mid \rho \text{ is a C}^*\text{-norm}\} = \infty$ and thus the universal C*-enveloping algebra of B does not exist. ⌟

Let us now take a look at the most important examples of compact matrix quantum groups.

Classical matrix quantum groups. We have already seen that we can assume that all compact matrix groups are subgroups of the unitary group $U_n = \{M \in \mathbb{C}^{n\times n} \mid M^*M = MM^* = I\}$ consisting of all $n \times n$ unitary matrices. The orthogonal group $O_n = \{M \in \mathbb{C}^{n\times n} \mid \overline{M} = M, M^\mathsf{T}M = MM^\mathsf{T} = I\}$ is given by all orthogonal $n \times n$ matrices. The symmetric group S_n of the permutations of an n-element set is also a compact matrix group, since the corresponding permutation matrices define a faithful matrix representation. Consequently, these groups form compact matrix quantum groups, and one can show that

$$C(U_n) \cong C^*\langle u_{ij}, 1 \le i,j \le n \mid \sum_{k=1}^n u_{ki}^* u_{kj} = \sum_{k=1}^n u_{ik}u_{jk}^* = \delta_{ij}\mathbf{1}, u_{ij}u_{kl} = u_{kl}u_{ij}\rangle$$
$$C(O_n) \cong C^*\langle u_{ij}, 1 \le i,j \le n \mid \sum_{k=1}^n u_{ik} u_{jk} = \sum_{k=1}^n u_{ki}u_{kj} = \delta_{ij}\mathbf{1}, u_{ij}^* = u_{ij}, u_{ij}u_{kl} = u_{kl}u_{ij}\rangle$$
$$C(S_n) \cong C^*\langle u_{ij}, 1 \le i,j \le n \mid \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = \mathbf{1}, u_{ij}^2 = u_{ij}^* = u_{ij}, u_{ij}u_{kl} = u_{kl}u_{ij}\rangle.$$

Liberated matrix quantum groups. We obtain quantum versions of U_n, O_n, S_n by dropping the commutativity constraint from the above relations. This procedure is sometimes called *liberation* and yields the *quantum unitary group* $U_n^+ := (C(U_n^+), (u_{ij}), n)$, the *quantum orthogonal group* $O_n^+ := (C(O_n^+), (u_{ij}), n)$, and the *quantum symmetric group* $S_n^+ := (C(S_n^+), (u_{ij}), n)$; given by

$$C(U_n^+) := C^*\langle u_{ij}, 1 \le i,j \le n \mid \sum_{k=1}^n u_{ki}^* u_{kj} = \sum_{k=1}^n u_{ik}u_{jk}^* = \delta_{ij}\mathbf{1}, \sum_{k=1}^n u_{ik}^* u_{jk} = \sum_{k=1}^n u_{ki}u_{kj}^* = \delta_{ij}\mathbf{1}\rangle$$
$$C(O_n^+) := C^*\langle u_{ij}, 1 \le i,j \le n \mid \sum_{k=1}^n u_{ik} u_{jk} = \sum_{k=1}^n u_{ki}u_{kj} = \delta_{ij}\mathbf{1}, u_{ij}^* = u_{ij}\rangle$$
$$C(S_n^+) := C^*\langle u_{ij}, 1 \le i,j \le n \mid \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = \mathbf{1}, u_{ij}^2 = u_{ij}^* = u_{ij}\rangle.$$

Note that for $C(U_n^+)$ we additionally had to require that $(u_{ij})^\mathsf{T}$ is invertible, since this only follows from the commutativity of the matrix elements in the classical case, but is required for U_n^+ to be a compact matrix quantum group. Finally, it is not hard to verify that

$$\begin{array}{ccccc} S_n^+ & \subseteq & O_n^+ & \subseteq & U_n^+ \\ \cup\mathrm{I} & & \cup\mathrm{I} & & \cup\mathrm{I} \\ S_n & \subseteq & O_n & \subseteq & U_n \end{array}.$$

In the following chapters, we will mostly be interested in compact matrix quantum groups between S_n and O_n^+.

Definition 3.40. We call a compact matrix quantum group G *orthogonal* if $G \subseteq O_n^+$.

Remark 3.41. Since the fundamental representation of a compact matrix quantum group has to be unitary, it is not hard to see that $G = (C(C), (u_{ij}), n)$ is orthogonal if and only if $u_{ij}^* = u_{ij}$ for all $i, j \in [n]$.

In the spirit of these definitions, we also define "quantum versions" of matrices. These matrices have entries in some C*-algebra and satisfy the same relations as the fundamental representations of their corresponding compact matrix quantum groups.

Definition 3.42. Let A be a C*-algebra. A matrix $\mathcal{U} \in A^{n\times n}$ is called a *quantum permutation* (*orthogonal*, *unitary*) *matrix* if its elements satisfy the defining relations of the quantum symmetric (orthogonal, unitary) group.

Quantum permutation matrices are also sometimes called *magic unitaries*, cf. [19]. Observe that if the C*-algebra is $\mathbb{C}$, we just recover the conventional notions of permutation, orthogonal, and unitary matrices.

3.3 Group Actions and Orbits

The way groups describe symmetries is through *group actions*. The idea is that the set of symmetries or automorphisms of an object form a group under function composition. As it turns out, very different objects can have isomorphic symmetry/automorphism groups. It thus makes sense to instead consider them as one abstract group that can "act on" objects. A pertinent example is the symmetric group S_n, which acts on arbitrary n-element sets by permuting its elements.

Definition 3.43. Let G be a topological group with identity element e. A *group action* on a topological space X is a continuous function $\alpha\colon X \times G \to X$ satisfying

(i) $\alpha(\alpha(h, x), g) = \alpha(x, gh)$

(ii) $\alpha(x, e) = x$

for all $g, h \in G$ and $x \in X$.

Example 3.44. Consider the circle group $\mathbb{S}^1$ of complex numbers with absolute value 1, introduced in Example 3.19. It describes the symmetries of a circle: For a circle in the complex plane centered at the origin, a group action is simply given by complex multiplication. ⌟

Example 3.45. As a discrete analogue of $\mathbb{S}^1$, the cyclic group C_n consisting of the n-th roots of unity describes the rotational symmetries of a regular n-sided polygon. For a regular polygon in the complex plane centered at the origin, a group action is again given by complex multiplication. It also describes the rotational symmetries of general polygons: For any m-sided polygon there exists a $k \leq m$ such that the group of its rotational symmetries is isomorphic to C_k. ⌟

We now want to derive an analogous concept for quantum groups. To that end, we need to express properties (i) and (ii) of Definition 3.43 in terms of the comultiplication. Recall that the comultiplication is a *dualisation* of the multiplication. A natural approach is thus to attempt to dualise Definition 3.43 in the same way by reasoning about the space of complex-valued functions over the topological space X. As such, a group action of a quantum group G will be a function $\hat{\alpha}\colon C(X) \to C(X) \otimes C(G)$. We can express property (i) as the following commutative diagram.

$$\begin{array}{ccc} X \times G \times G & \xrightarrow{\alpha \times \mathrm{id}} & X \times G \\ \downarrow{\scriptstyle \mathrm{id} \times \cdot} & & \downarrow{\scriptstyle \alpha} \\ X \times G & \xrightarrow{\alpha} & X \end{array}$$

Dualising this yields

$$\begin{array}{ccc} C(X) \otimes C(G) \otimes C(G) & \xleftarrow[\hat{\alpha} \otimes \mathrm{id}]{} & C(X) \otimes C(G) \\ {\scriptstyle \mathrm{id} \otimes \Delta}\uparrow & & \uparrow{\scriptstyle \hat{\alpha}} \\ C(X) \otimes C(G) & \xleftarrow[\hat{\alpha}]{} & C(X) \end{array}$$

For classical groups with group action α, one can verify that this diagram commutes for $\hat{\alpha}$ given by

$$\hat{\alpha}\colon C(X) \to C(X) \otimes C(G), f \mapsto [X \times G \to \mathbb{C}, (x,g) \mapsto f(\alpha(x,g))]. \tag{3.2}$$

Property (ii) is harder to dualise because we lack a counit. Fortunately, a similar trick as in our definition of compact quantum groups can be applied. First note that we obtain an equivalent definition of group actions by replacing property (ii) with

(ii') $\alpha(x,g) = \alpha(y,g) \Longrightarrow x = y$ for all $g \in G$.

Indeed, we have $\alpha(\alpha(x,e),e) = \alpha(x, e \cdot e) = \alpha(x,e)$, which by (ii') implies $\alpha(x,e) = x$. Conversely, we have $\alpha(x,g) = \alpha(y,g) \Rightarrow \alpha(\alpha(x,g),g^{-1}) = \alpha(\alpha(y,g),g^{-1}) \Rightarrow \alpha(x,e) = \alpha(y,e) \Rightarrow x = y$. We can then prove the the following analogue of Lemma 3.23.

Lemma 3.46. *Let G be a compact group, X a topological space, and $\alpha\colon X \times G \to G$. Let $\hat{\alpha}$ be defined as in Equation* (3.2)*. Then α satisfies property (ii') if $\hat{\alpha}(C(X))(\mathbf{1} \otimes C(G))$ is linearly dense in $C(X) \otimes C(G)$.*

Proof. Let $x, y \in X$, $g \in G$ such that $\alpha(x,g) = \alpha(y,g)$. Then we have

$$(\hat{\alpha}(\lambda)(\mathbf{1} \otimes \kappa))(x,g) = \lambda(\alpha(x,g)) \cdot \kappa(g) = \lambda(\alpha(y,g)) \cdot \kappa(g) = ((\mathbf{1} \otimes \kappa)\hat{\alpha}(\lambda))(y,g)$$

for all $\kappa \in C(G)$, $\lambda \in C(X)$. Since $\hat{\alpha}(C(X))(\mathbf{1} \otimes C(G))$ is dense in $C(X) \otimes C(G)$, we have $h(x,g) = h(y,g)$ for all $h \in C(X) \otimes C(G)$ and hence by Urysohn's lemma $x = y$ as desired. □

In particular, classical groups with action given by Equation (3.2) satisfy this property, as one can show along similar lines as in Lemma 3.24.

Lemma 3.47. *Let G be a compact group, X a topological space, and let $\hat{\alpha}$ be defined as in Equation* (3.2)*. Then $\hat{\alpha}(C(X))(\mathbf{1} \otimes C(G))$ is linearly dense in $C(X) \otimes C(G)$.*

Proof. Define the mapping $h\colon X \times G \to X \times G, (x,g) \mapsto (\alpha(x,g),g)$. Then h is a homeomorphism: We show that h is bijective and a continuous open mapping. The injectivity follows immediately from property (ii'). If $h(x,g) = (\alpha(x,g),g) = (\alpha(y,g),g) = h(y,g)$, then $\alpha(x,g) = \alpha(y,g)$ and hence $x = y$. To show that h is surjective, consider some $(x,g) \in X \times G$. Then $h(\alpha(x,g^{-1}),g) = (\alpha(\alpha(x,g),g^{-1}),g) = (\alpha(x,gg^{-1}),g) = (x,g)$. Finally, h is continuous because α is continuous. For the same reason, the inverse $h^{-1}\colon (x,g) \mapsto (\alpha(x,g^{-1}),g)$ of h is also continous, hence h is an open mapping.

In particular, this means that $\kappa \otimes \lambda \mapsto (\kappa \otimes \lambda) \circ h$ is an automorphism of $C(X) \otimes C(G)$. Concretely, we have $(\kappa \otimes \lambda) \circ h = [(x,g) \mapsto \kappa(\alpha(x,g)) \cdot \lambda(g)]$. But this is just equal to $\hat{\alpha}(\kappa)(1 \otimes \lambda)$, hence $\hat{\alpha}(C(X))(1 \otimes C(G))$ is linearly dense in $C(X) \otimes C(G)$. □

Consequently, we define the action of a compact quantum group as follows.

Definition 3.48. An *action* of a compact quantum group G on a topological space X is a *-homomorphism $\hat{\alpha}\colon C(X) \to C(X) \otimes C(G)$ satisfying

(i) $(\hat{\alpha} \otimes \mathrm{id}) \circ \hat{\alpha} = (\mathrm{id} \otimes \Delta) \circ \hat{\alpha}$

(ii) $\hat{\alpha}(C(X))(1 \otimes C(G))$ is linearly dense in $C(X) \otimes C(G)$.

3.3.1 Quantum Orbits

For classical groups G with group action $\alpha\colon X \times G \to X$, the orbit of an element $x \in X$ is the set $G \cdot x := \{\alpha(x,g) \mid g \in G\}$ of elements that x can be moved to by elements of G. The *orbit space* is thus given by the quotient $X/G := \{\alpha(G,x) \mid x \in X\}$.

To translate this concept to the quantum case, we again consider the continuous complex-valued functions on X/G, which we can identify with the subalgebra $C(X)^G$ of $C(X)$ consisting of the functions $f \in C(X)$ that are constant on the orbits of G. Now for the (quantum) action $\hat{\alpha}\colon C(X) \to C(X) \otimes C(G)$ given by Equation (3.2), $f \in C(X)$ is constant on the orbits of G if and only if

$$\hat{\alpha}(f) = [(x,g) \mapsto f(\alpha(x,g))] = [(x,g) \mapsto f(x)] = f \otimes \mathbf{1}.$$

Definition 3.49. Let G be a compact quantum group, X a topological space, and $\hat{\alpha}$ a group action of G. The quantum orbit space X/G is given by the C*-subalgebra

$$C(X/G) = \{f \in C(X) \mid \hat{\alpha}(f) = f \otimes \mathbf{1}\}.$$

In this thesis we are mostly interested in groups that act on graphs, so X is a finite set of vertices (with the discrete topology). In this case $C(X) \cong \mathbb{C}^X$. Moreover, the quantum symmetric group enjoys a particularly nice characterisation. Firstly, S_n^+ admits a canonical action $\hat{\alpha}_u$ given by

$$\hat{\alpha}_u\colon \mathbb{C}^X \to \mathbb{C}^X \otimes C(S_n^+), e_x \mapsto \sum_y e_y \otimes u_{yx},$$

where e_x denotes the canonical basis of $\mathbb{C}^X$ and (u_{ij}) is the fundamental representation of S_n^+. We can view this as the analogue to the canonical action of S_n, which permutes the elements of a vector $v \in \mathbb{C}^X$. Lupini, Mančinska, and Roberson [19] showed that two elements $x, y \in X$ are in the same orbit of S_n^+ with respect to this action if and only if $u_{xy} \neq 0$.

Theorem 3.50 (Lemma 3.7 in [19])**.** *Let X be a set of size n and $(C(G), (u_{ij}), n) \subseteq S_n^+$ be a compact matrix quantum subgroup of S_n^+ with canonical action $\hat{\alpha}_u$. Then for all $f \in \mathbb{C}^X$, we have $\hat{\alpha}_u(f) = f \otimes \mathbf{1}$ if and only if $f(x) = f(y)$ for all $x, y \in X$ with $u_{xy} \neq 0$.*

3.4 Classifying Easy Quantum Groups

A central result in the theory of compact quantum groups, and subject of our study in the later chapters, is the full classification of easy quantum groups. Easy quantum groups are a subclass of compact matrix quantum groups that can be described entirely combinatorically through so called *partition categories*, that turn out to describe the *intertwiners* of the group. In this section, we want to understand this result, starting with the representation theory of compact quantum groups.

3.4.1 Representations & Intertwiners

Representation theory is an important area in the study of abstract algebraic structures. The idea is to represent the elements of these structures as linear transformations of vector spaces. The hope is then to gain more insight into the structures using the powerful tools of linear algebra. In the line of what we have seen so far, it comes as no surprise that we can also develop a representation theory for quantum groups.

Definition 3.51. Let G be a compact group. A *matrix representation* of G is a continuous group homomorphism $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

Analogous to how we introduced compact matrix quantum groups, we can view such a matrix representation ρ as a matrix $\mathcal{V} = (v_{ij})$ over the C*-algebra of continuous complex-valued functions $C(G)$ by letting

$$v_{ij}\colon G \to \mathbb{C}, x \mapsto \rho(x)_{ij}.$$

These functions are thus simply elements of the underlying C*-algebra of G as a quantum group, and we have

$$\Delta(v_{ij})(x,y) = v_{ij}(xy) = \rho(xy)_{ij} = \sum_k \rho(x)_{ik}\rho(y)_{kj} = \sum_k v_{ik}(x)v_{kj}(y),$$

or under the isomorphism $C(G \times G) \cong C(G) \otimes C(G)$,

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}. \tag{3.3}$$

Analogously to compact matrix quantum groups, we can recover a matrix representation from every matrix that has entries in a commutative C*-algebra and satisfies Equation (3.3), justifying the following definition.

Definition 3.52. Let $G = (C(G), \Delta)$ be a compact quantum group. A *finite-dimensional representation* of G is a matrix $\mathcal{V} = (v_{ij}) \in C(G)^{m\times m}$ for some $m \in \mathbb{N}$, satisfying

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}.$$

Remark 3.53. Note that the fundamental representation $\mathcal{U}$ of a compact matrix quantum group $(C(G), \mathcal{U}, n)$ is a finite-dimensional representation in this sense.

Finite-dimensional representations inherit certain properties from their underlying matrix structure. As such, we can define the *direct sum* $\mathcal{U} \oplus \mathcal{V}$ of two finite-dimensional representations $\mathcal{U} = (u_{ij}) \in C(G)^{m\times m}, \mathcal{V} = (v_{ij}) \in C(G)^{m'\times m'}$ as the $(m+m') \times (m+m')$ matrix

$$\begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{V} \end{pmatrix},$$

and the *tensor product* $\mathcal{U} \otimes \mathcal{V}$ as the Kronecker product, that is, the $mm' \times mm'$ matrix whose (i_1i_2, j_1j_2) entry is $u_{i_1j_1} \cdot v_{i_2j_2}$. We also write $\mathcal{U}^{\otimes k}$ for the k-fold tensor product of $\mathcal{U}$ and let $\mathcal{U}^{\otimes 0} = (\mathbf{1})$. In line with fundamental representations of compact matrix quantum groups, we call $\mathcal{U}$ *unitary* if $\mathcal{U}^*\mathcal{U} = \mathcal{U}\mathcal{U}^* = \mathbf{1}I$, where $(u_{ij})^* = (u_{ji}^*)$. We call $\mathcal{U}$ *invertible* if there exists some $\mathcal{U}^{-1}$ with $\mathcal{U}\mathcal{U}^{-1} = \mathcal{U}^{-1}\mathcal{U} = \mathbf{1}I$. Finally, we call $\mathcal{U}$ *reducible* if there exists a $T \in \mathrm{GL}_m(\mathbb{C})$ such that

$$T\mathcal{U}T^{-1} = \begin{pmatrix} \mathcal{U}_1 & \mathcal{U}_2 \\ 0 & \mathcal{U}_3 \end{pmatrix},$$

for some matrices $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$.

We usually want to study how different representations are related, the most important notion to that end being that of an intertwiner.

Definition 3.54. An *intertwiner* is an equivariant map between two finite-dimensional representations $\mathcal{U} \in C(G)^{m\times m}$, $\mathcal{V} \in C(G)^{m'\times m'}$, that is, a linear map $T\colon \mathbb{C}^m \to \mathbb{C}^{m'}$ such that $T\mathcal{U} = \mathcal{V}T$.

Here we view T as a matrix $\mathbb{C}^{m'\times m} \subseteq C(G)^{m'\times m}$ and do matrix multiplication over $C(G)$.

Definition 3.55. Two finite-dimensional representations $\mathcal{U}, \mathcal{V} \in C(G)^{m\times m}$ are called *equivalent* if there exists an invertible intertwiner between them.

These definitions are often phrased in category theoretic language. Namely, the space of finite-dimensional representations of a compact quantum group G forms the *representation category* $\mathrm{Rep}(G)$, which is given by

Objects:	finite-dimensional unitary representations of G,
Morphisms:	$\mathrm{Mor}(\mathcal{V},\mathcal{V}') = \{T\colon \mathbb{C}^m \to \mathbb{C}^{m'} \text{ linear} \mid T\mathcal{V} = \mathcal{V}'T\}$,
Composition:	natural composition.

In the case of orthogonal compact matrix quantum groups, our main objects of study, this category takes a particularly convenient form.

Theorem 3.56 (cf. [20, Theorem 2.1.9])**.** *Let $G = (C(G),\mathcal{U},n)$ be an orthogonal compact matrix quantum group. Then*

(*i*) *Every invertible finite-dimensional representation G is equivalent to a unitary representation of G.*

(*ii*) *Every unitary representation of G decomposes into a direct sum of irreducible representations of G.*

(*iii*) *Every unitary irreducible representation of G is either the trivial representation $(\mathbf{1})$ or equivalent to a direct summand of a tensor power of $\mathcal{U}$.*

In this sense, the entire structure of the space of finite-dimensional representations is already contained in the tensor powers of the fundamental representation $\mathcal{U}$.[2] For this reason, we define the *reduced representation category* of a compact matrix quantum group $(C(G),\mathcal{U},n)$ as the category $\mathrm{Rep}_0(G)$ given by

Objects:	$\mathcal{U}^{\otimes m}$, $\forall m \in \mathbb{N}$,
Morphisms:	$\mathrm{Mor}(\mathcal{U}^{\otimes k},\mathcal{U}^{\otimes \ell}) = \{T\colon (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes \ell} \text{ linear} \mid T\mathcal{U}^{\otimes k} = \mathcal{U}^{\otimes \ell}T\}$,
Composition:	natural composition.

One can now verify that the morphisms of $\mathrm{Rep}_0(G)$ form a *concrete tensor category with duals*.

Definition 3.57. Let $n \in \mathbb{N}$. A *concrete tensor category with duals* C is a collection of $\mathbb{C}$-vector spaces $C(\ell,k) \subseteq \{T\colon (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes \ell} \text{ linear}\}$ for all $k,\ell \in \mathbb{N}$, such that

(i) if $T \in C(\ell,k)$ and $T' \in C(r,s)$, then $T \otimes T' \in C(\ell+r, k+s)$,

(ii) if $T \in C(\ell,k)$ and $T' \in C(k,r)$, then $TT' \in C(\ell,r)$,

(iii) if $T \in C(\ell,k)$, then $T^* \in C(k,\ell)$,

(iv) $I \in C(1,1)$,

(v) $\xi = \sum_i e_i \otimes e_i \in C(2,0)$.

[2] One can make this statement more precise. In fact, we will see later that the category of all finite-dimensional representations is the so called pseudoabelian completion of the space of tensor powers of $\mathcal{U}$. For a more in-depth treatment, refer to [20, Chapter 2].

Remark 3.58. The name "concrete tensor categories with duals" stems from the fact that these objects represent so called C*-tensor categories with duals. These categories are a subclass of *rigid monoidal categories*. A *monoidal category* is a category $\mathcal{C}$ together with a binary map $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and a distinguished unit object $\mathbf{1}\in \mathrm{Obj}(\mathcal{C})$ that induce a monoid structure on the objects. It is called *rigid* if every object has a *dual*, that is, an object satisfying certain conjugacy relations with respect to $\otimes$. Concretely, the existence of the map ξ guarantees that the objects corresponding to $(\mathbb{C}^n)^{\otimes 1}$ are self-dual.

Woronowicz showed that there is a one-to-one correspondence between concrete tensor categories with duals and compact matrix quantum groups. We will use the following version of this duality, formulated by Maaßen [20, Corollary 2.4.6].

Theorem 3.59 (Woronowicz' Tannaka-Krein duality)**.** *Let $n\in\mathbb{N}$ and C be a concrete tensor category with duals. Then there exists a compact matrix quantum group $(C(G),\mathcal{U},n)$ with*

$$\mathrm{Mor}_{\mathrm{Rep}_0(G)}(\mathcal{U}^{\otimes k},\mathcal{U}^{\otimes \ell}) = C(\ell,k)$$

for all $k,\ell\in\mathbb{N}$. Moreover, G is unique up to similarity.

In this sense, compact matrix quantum groups are uniquely determined by the morphisms of their representation categories, which are precisely the intertwiners of tensor powers of the fundamental representation. For this reason, it suffices to use the following simplified notation.

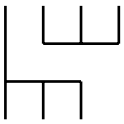
Definition 3.60. Let $(C(G),\mathcal{U},n)$ be a compact matrix quantum group. We denote by C^G the concrete tensor category with duals formed by the morphisms of $\mathrm{Rep}_0(G)$ and refer to it as the *intertwiners of G*. Similarly, we refer to the sets $C^G(\ell,k)$ as the *(ℓ,k)-intertwiners of G*.

3.4.2 Partition Categories

It turns out that concrete tensor categories with duals arise naturally in certain contexts. An important example are categories of partitions. Concretely, we are interested in partitions of a set $\{1_L,\dots,\ell_L,1_U,\dots,k_U\}$, where we think of $1_L,\dots,\ell_L$ as the *lower points*, and of $1_U,\dots,k_U$ as the *upper points* of the partition. We denote the set of these partitions by $\mathbb{P}(\ell,k)$. We can define such a partition graphically by drawing the upper points above the lower points and connecting points that are in the same block of the partition, as shown in Figure 3.1.



$P_1 = \{\{1_L\},\{1_U,2_U,2_L\},\{3_U,4_L\},\{4_U,3_L\}\}$



$P_2 = \{\{1_U,1_L,2_L,3_L\},\{2_U,3_U,4_U\}\}$

Figure 3.1: Drawing partitions

An important subclass are *non-crossing* partitions, which are the partitions that can be drawn without any lines crossing.

Definition 3.61. A partition P of an ordered set V is called *non-crossing* if whenever $a<b<c<d$, and a,c are in the same part and b,d are in the same part, then the two parts coincide. We define non-crossing partitions of $\{1_L,\dots,\ell_L,1_U,\dots,k_U\}$ by letting $1_L<\dots<\ell_L<1_U<\dots<k_U$.

We can now define operations on partitions. Let $P\in\mathbb{P}(\ell,k)$ and $P'\in\mathbb{P}(\ell',k')$. Then the tensor product $P\otimes P'\in P(\ell+\ell',k+k')$ is the partition obtained by drawing P and P' next to each other horizontally, as shown in Figure 3.2. If $\ell'=k$, then the composition $P'\circ P\in\mathbb{P}(k',\ell)$

is obtained by drawing P' below P and identifying the lower row of P and the upper row of P'. Note that this operation may create *loops*, i.e. blocks that are no longer connected to the upper or lower row of the resulting partition. These become *empty blocks* of $P' \circ P$. Finally, the *adjoint* $P^* \in \mathbb{P}(k, \ell)$ is obtained by reflecting P across the horizontal axis.

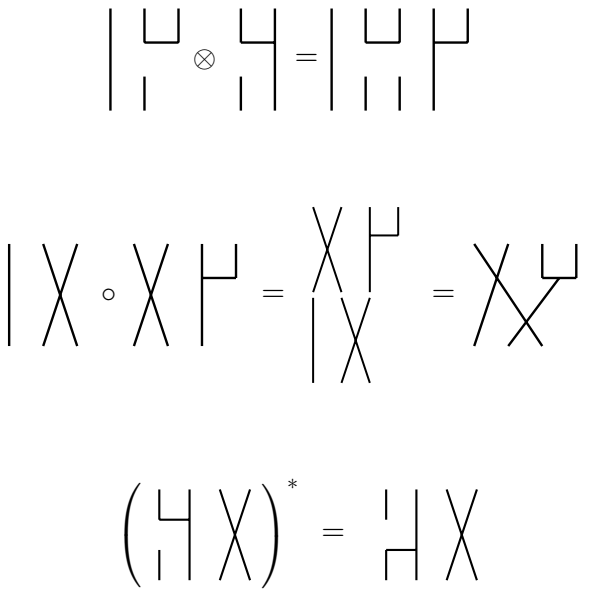


Figure 3.2: Operations on partitions

To any partition $P \in \mathbb{P}(\ell, k)$ and $n \geq 1$, we can associate a linear map $T_P^{(n)} \colon (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes \ell}$ as follows. For $\mathbf{a} \in [n]^\ell, \mathbf{b} \in [n]^k$, we let $\delta(\mathbf{a}, \mathbf{b}) = 1$ if the assignments $i_U \mapsto b_i$ and $i_L \mapsto a_i$ are constant on the blocks of P, and $\delta(\mathbf{a}, \mathbf{b}) = 0$ otherwise. Then $T_P^{(n)}$ is defined on basis vectors $e_{b_1} \otimes \cdots \otimes e_{b_k} \in (\mathbb{C}^n)^{\otimes k}$ as

$$T_P^{(n)}(e_{b_1} \otimes \cdots \otimes e_{b_k}) = n^{e(P)} \sum_{\mathbf{a} \in [n]^\ell} \delta(\mathbf{a}, \mathbf{b})(e_{a_1} \otimes \cdots \otimes e_{a_\ell}),$$

where $e(P)$ denotes the number of empty blocks in P. This definition has a nice graphical interpretation, where we simply shuffle the input vectors according to the blocks, see Figure 3.3. This suggest to write the matrix form of $T_P^{(n)}$ in terms of outer products: Consider, for instance, the partition P_1 in Figure 3.1. The corresponding matrix can be seen to be equal to

$$T_{P_1}^{(n)} = \sum_{ijk\ell} (e_\ell \otimes e_i \otimes e_k \otimes e_j) \cdot (e_i \otimes e_i \otimes e_j \otimes e_k)^\mathsf{T}.$$

Before we continue, we want to highlight some important partions that we will refer to regularly. The first group are the single block partitions $\{\{1_L, \ldots, \ell_L, 1_U, \ldots k_U\}\} \in \mathbb{P}(\ell, k)$, the corresponding linear maps map k copies of a vector to ℓ copies. We call these maps the *generalised multiplication maps* and denote them by $M^{(n),\ell,k}$. Two important special cases are $M^{(n),1,1} = I_n$ and $M^{(n),2,0} = \sum_i e_i \otimes e_i = \xi$. The second important partition is the *swap* partition $\times = \{\{1_U, 2_L\}, \{2_U, 1_L\}\}$. Its corresponding linear map is $S^{(n)} \colon e_i \otimes e_j \mapsto e_j \otimes e_i$. From now on, we will omit the n in the superscript if it is clear from the context.

Banica and Speicher [3] showed that the operations of tensor product, composition, and adjoint on partitions coincide with the corresponding operations on linear maps.

Proposition 3.62. *Let $P, P' \in \mathbb{P}$. Then we have $T_{P \otimes P'} = T_P \otimes T_{P'}$, $T_{P^*} = T_P^*$, and if P and P' are compatible, $T_{P \circ P'} = T_P \circ T_{P'}$.*

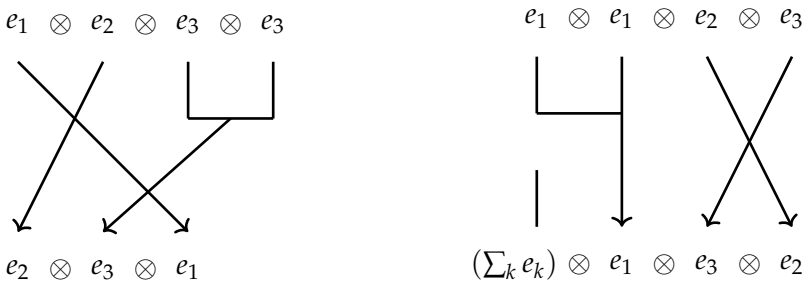


Figure 3.3: Partitions as linear maps

This allows us to define *partition categories* that give rise to concrete tensor categories with duals.

Definition 3.63. A *partition category* $\mathcal{C}$ is a set of partitions, such that

(i) if $P \in \mathcal{C}(\ell, k)$ and $P' \in \mathcal{C}(r, s)$, then $P \otimes P' \in \mathcal{C}(\ell + r, k + s)$,

(ii) if $P \in \mathcal{C}(\ell, k)$ and $P' \in \mathcal{C}(k, r)$, then $P \circ P' \in \mathcal{C}(\ell, r)$,

(iii) if $P \in \mathcal{C}(\ell, k)$, then $P^* \in \mathcal{C}(\ell, r)$,

(iv) $\{\{1_U, 1_L\}\} = \, | \, \in \mathcal{C}(1, 1)$,

(v) $\{\{1_L, 2_L\}\} = \sqcap \in \mathcal{C}(2, 0)$,

where $\mathcal{C}(\ell, k) = \mathcal{C} \cap \mathbb{P}(\ell, k)$.

For partitions $P_1, \ldots, P_m \in \mathbb{P}$, we write $\langle P_1, \ldots, P_m \rangle$ for the partition category generated by the P_i, that is, the closure of $\{P_1, \ldots, P_m, \sqcap, | \}$ under tensor products, composition, and taking adjoints. Now it is easy to see that for every partition category $\mathcal{C}$, the collection of vector spaces

$$C(\ell, k) = \operatorname{span}\{T_P \mid P \in \mathcal{C}(\ell, k)\}$$

forms a concrete tensor category with duals. Theorem 3.59 thus implies the following result.

Proposition 3.64. *Let $\mathcal{C}$ be a partition category. Then for each $n \geq 1$ there exists a unique compact matrix quantum group $G_n(\mathcal{C}) = (A, \mathcal{U}, n)$ such that*

$$\mathcal{U}^{\otimes k} T_P = T_P \mathcal{U}^{\otimes \ell}$$

for all $P \in \mathcal{C}(k, \ell)$.

We call compact matrix quantum groups of this form *easy quantum groups*. Moreover, one can prove—in fact we will in Chapter 5—that $\mathcal{U}^{\otimes 2} S = S \mathcal{U}^{\otimes 2}$ if and only if the entries of $\mathcal{U}$ commute. Hence, an easy quantum group $G_n(\mathcal{C})$ is classical if and only if $\times \in \mathcal{C}$.

3.4.3 The Full Classification

We can now state the full classification of partition categories and thus of easy quantum groups, due to Raum and Weber [26]. We follow the presentation of Maaßen [20, Section 3.1.5].

Theorem 3.65 (Classification of Partition Categories)**.** *Every category of partition falls into one of the following cases.*

(*i*) *The partition categories $\mathcal{C}$ with $\times \in \mathcal{C}$ are exactly*

- $P = \langle \times, \uparrow, \sqcap\!\!\sqcup \rangle$*, all partitions,*

- $P_2 := \langle \diagup\!\diagdown \rangle$, *the partitions with block size two,*
- $P_{even} := \langle \diagup\!\diagdown, \sqcap\!\sqcap \rangle$, *the partitions with even block size,*
- $P' := \langle \diagup\!\diagdown, \uparrow \otimes \uparrow, \sqcap\!\sqcap \rangle$, *the partitions with an even number of blocks of odd size,*
- $P_b := \langle \diagup\!\diagdown, \uparrow \rangle$, *the partitions with block size one or two,*
- $P'_b := \langle \diagup\!\diagdown, \uparrow \otimes \uparrow \rangle = \langle \diagup\!\diagdown, \mid\!\diagdown\!\mid \rangle$, *the partitions with an arbitrary number of blocks of size two and an even number of blocks of size one.*

(*ii*) *The partition categories that contain only non-crossing partitions are exactly*

- $NC = \langle \uparrow, \sqcap \rangle$, *all non-crossing partitions,*
- $NC_2 := \langle \varnothing \rangle = \langle \mid, \sqcap \rangle$, *the non-crossing partitions with block size two,*
- $NC_{even} := \langle \sqcap\!\sqcap \rangle$, *the non-crossing partitions with even block size,*
- $NC' := \langle \uparrow \otimes \uparrow, \sqcap\!\sqcap \rangle$, *the non-crossing partitions with an even number of blocks of odd size,*
- $NC_b := \langle \uparrow \rangle$, *the non-crossing partitions with block size one or two,*
- $NC'_b := \langle \mid\!\diagdown\!\mid \rangle$, *the non-crossing partitions with an arbitrary number of blocks of size two and an even number of blocks of size one.*
- $NC_b^{\#} := \langle \uparrow \otimes \uparrow \rangle$, *the partitions with an arbitrary number of blocks of size two, each connecting an odd and an even point, and an even number of blocks of size one.*

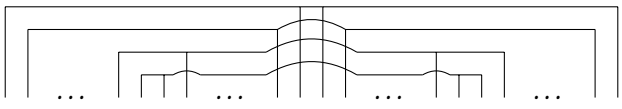
(*iii*) *The partition categories $\mathcal{C}$ with $\diagup\!\diagdown \notin \mathcal{C}$ and $\times \in \mathcal{C}$ are exactly*

- $E_o := \langle \times \rangle$, *the partitions with blocks of size two, each connecting an odd and an even point,*
- $E'_b := \langle \times, \uparrow \otimes \uparrow \rangle$, *the partitions with an arbitrary number of blocks of size two—each connecting an odd and an even point, and an even number of blocks of size one,*
- $E_h := \langle \times, \sqcap\!\sqcap \rangle$, *the partitions with each block having the same number of odd and even points,*
- $E_h^s := \langle \times, \sqcap\!\sqcap, h_s \rangle$, *where $h_s = \{\{1, 3, \ldots, (2s-1)\}, \{2, 4, \ldots, 2s\}\} \in P(2s, 0)$, the partitions with an even number of points and each block having the same number of odd and even points modulo s for all $s \geq 3$.*

(*iv*) *The partition categories $\mathcal{C}$ with $\diagup\!\diagdown\!\sqcap\!\sqcap \in \mathcal{C}$ are called* group-theoretic *and are indexed by certain subgroups of $\mathbb{Z}_2^{*\infty}$.*

(*v*) *The partition categories $\mathcal{C}$ with $\sqcap\!\sqcap \in \mathcal{C}$, $\uparrow \otimes \uparrow \notin \mathcal{C}$ and $\diagup\!\diagdown\!\sqcap\!\sqcap \notin \mathcal{C}$ are exactly the categories $\Pi_k = \langle \pi_k \rangle$, generated by the element*

$$\pi_k = \quad \in \mathbb{P}(0, 4k)$$



and $\Pi_\infty = \langle \pi_k \mid k \in \mathbb{N} \rangle$.

The cases in (i), (ii), (iii) are pairwise distinct.

We will mainly focus on the partition categories in (i), (ii), and (iii). As noted above, these partition categories form tensor categories with duals and thus uniquely correspond to compact matrix quantum groups. Moreover, the compact matrix quantum groups with intertwiner $\diagup\!\diagdown$ have commuting C*-algebras and thus correspond to classical groups.

Proposition 3.66. *The easy groups, i.e. the easy quantum groups that correspond to partition categories $\mathcal{C}$ with $\diagup\!\diagdown \in \mathcal{C}$ are*

(*i*) $G_n(P) = S_n$, *the symmetric group,*

(*ii*) $G_n(P_2) = O_n$, *the orthogonal group,*

(*iii*) $G_n(P_{even}) = H_n$, *the hyperoctahedral group, i.e. the group of monomial orthogonal matrices,*

(*iv*) $G_n(P') = S'_n$, *the modified symmetric group, i.e. the group of signed permutation matrices,*

(*v*) $G_n(P_b) = B_n$, *the bistochastic group, i.e. the group of orthogonal matrices having row and column sum* 1,

(*vi*) $G_n(P'_b) = B'_n$, *the modified bistochastic group, i.e. the group of signed bistochastic matrices.*

Here a matrix is called *monomial* if it has the same zero pattern as a permutation matrix, and a signed permutation (bistochastic) matrix is a permutation (bistochastic) matrix multiplied by ± 1. Analogously to Section 3.2.4, we obtain "liberated" quantum versions of these groups by dropping the commutativity constraint. In our case this corresponds to dropping the $\diagup\!\diagdown$ partition from the generating sets.

Proposition 3.67. *The easy quantum groups that correspond to categories of non-crossing partitions are*

(*i*) $G_n(NC) = S_n^+$, *the quantum symmetric group,*

(*ii*) $G_n(NC_2) = O_n^+$, *the quantum orthogonal group,*

(*iii*) $G_n(NC_{even}) = H_n^+$, *the quantum hyperoctahedral group,*

(*iv*) $G_n(NC') = S_n'^+$, *the quantum modified symmetric group,*

(*v*) $G_n(NC_b) = B_n^+$, *the quantum bistochastic group,*

(*vi*) $G_n(NC'_b) = B_n'^+$, *the quantum complexly modified bistochastic group,*

(*vii*) $G_n(NC_b^\#) = B_n^{\#+}$, *the quantum modified bistochastic group.*

An interesting special case occurs with the liberation of the modified bistochastic group. In the classical case, the group has two natural generating sets in $\langle \diagup\!\diagdown, \uparrow \otimes \uparrow \rangle$ and $\langle \diagup\!\diagdown, \sqcap\!\!\sqcup \rangle$. Dropping the commutativity constraint, however, yields the distinct easy quantum groups $B_n'^+$ and $B_n^{\#+}$ respectively.

Remark 3.68. In the literature on easy quantum groups, $B_n'^+$ and $B_n^{\#+}$ are usually called "quantum modified bistochastic group" and "quantum freely modified bistochastic group" respectively. However, as we will see later, the intertwiner $\uparrow \otimes \uparrow$ captures the notion of signed bistochastic matrices in a much more intuitive way than $\sqcap\!\!\sqcup$. It thus makes sense to instead call $B_n^{\#+}$ the quantum modified bistochastic group. The name "quantum complexly modified bistochastic group" is justified by the fact that $B_n'^+$ is the so called *tensor 2-complexification* of B_n^+, see [32, Remark 4.14].

3.5 Interlude: Why *quantum* groups?

Before we conclude this chapter, we want to take some time to justify the name "quantum groups". One could be satisfied with the reason that quantum groups have origins and applications in particle physics, which is inherently a quantum theory. However, the connection to quantum physics runs much deeper. We have seen that quantum groups are a non-commutative generalisation of classical groups, and in a precise sense non-commutativity is *the* defining feature of

quantum mechanics. In fact, we can derive quantum mechanics from classical mechanics in the same way we developed quantum groups from classical groups!

Consider a physical system of a single particle. In classical mechanics, we can describe the state of the system by a point (x, p) in some 2-dimensional space, describing the particle's position and momentum. In any real physical system, both the particles position and its momentum always remain bounded, and if the system can be arbitrarily close to a certain state, there is no reason why it should not be able to attain that precise state itself. It therefore makes sense to assume that the space of states is *compact*.

Now we would like to learn something about the system: we want to perform an experiment that measures some quantity of the system to an arbitrarily precise degree. We call these quantities *classical observables*. Examples are the position and momentum themselves, which we classically assume to be measureable to arbitrary precision. Mathematically, the observables should be functions of the system's state, i.e. x and p, and they should give us physically meaningful values, so they should be real-valued. The condition that an observable f should be measurable to an arbitrarily precise degree means that for every $\varepsilon > 0$ there should be some $\delta > 0$ such that measuring x and p with error at most δ and then plugging these approximations into f gives us an approximation of the value of the observable with error at most ε. In other words, f should be *continuous*.

Classical mechanics can be entirely described in terms of these two definitions. In quantum mechanics, however, the position and momentum of a particle are not fixed. Hence, if we want to have any hope of generalising this formalism to the quantum case, we need to abstract from the underlying space of possible states (x, p). In Section 3.1.2, we have already seen that the continuous complex-valued functions on a compact Hausdorff space form a commutative C*-algebra. Assuming our state space is Hausdorff, the classical observables are then exactly the self-adjoint elements of a commutative C*-algebra. This suggests to rephrase the above definitions purely in terms of C*-algebras. To define states in this framework, let us denote the space of valid classical states by X. For each $S = (x, p) \in X$, we define the evaluation map

$$\psi_S \colon C(X) \to \mathbb{C}, f \mapsto f(S).$$

Then one can show that ψ_S is a normalised positive linear functional on $C(X)$. Moreover, by the Riesz-Markov-Kakutani theorem, we can write every positive linear functional in terms of these evaluation maps.

Theorem 3.69 (Riesz-Markov-Kakutani)**.** *Let X be a compact Hausdorff space and let ψ be a normalised positive linear functional on $C(X)$. Then there exists a unique Borel probability measure μ_ψ on X, such that*

$$\psi(f) = \int_X \psi_S(f) d\mu_\psi(S).$$

We can therefore interpret every normalised positive linear functional ψ on $C(X)$ as a potentially *mixed state*, where we have some uncertainty about the state, and the system is in the *pure state* ψ_S with probability $\mu_\psi(S)$. We also interpret the value $\psi(f)$ as the *expected value* of the observable f in the state ψ. It thus makes sense to define the *variance* of an observable with respect to a state as

$$\sigma_\psi(f)^2 := \psi(f^2) - \psi(f)^2.$$

This is is classical mechanics.

Definition 3.70 (Classical Observables)**.** The observables of a *classical system* are the self-adjoint elements of a commutative C*-algebra A.

Definition 3.71 (Classical States)**.** The states of a *classical system* are the normalised positive linear functionals on A.

Now that we have formulated classical mechanics in terms of C*-algebras, we want to generalise this framework to describe quantum mechanics. Let us see what goes wrong with the current definitions. First note that for classical pure states ψ_S, we have $\sigma_{\psi_S}(f)^2 = 0$ for all observables f. It is however an *experimental fact* that for any quantum state φ we have

$$\sigma_\varphi(x)\sigma_\varphi(q) \geq \frac{\hbar}{2}. \tag{3.4}$$

Obviously we have to modify our definitions, and the following calculation suggests that the problem is the commutativity of observables. Consider two observables A, B and fix some state ψ such that $\psi(A) = \psi(B) = 0$. Thus

$$\sigma_\psi(A)^2\sigma_\psi(B)^2 = \psi(A^2)\psi(B^2).$$

Let $\alpha, \beta \in \mathbb{R}$ and consider $C = \alpha A + i\beta B$. Then C^*C is positive and hence

$$\begin{aligned}\psi(C^*C) &= \psi(\alpha^2A^2 + i\alpha\beta AB - i\alpha\beta BA + \beta^2B^2)\\ &= \alpha^2\psi(A^2) = \alpha\beta\psi(i[A,B]) + \beta^2\psi(B^2)\\ &\geq 0,\end{aligned}$$

where $[A, B] = AB - BA$ is the *commutator* of A and B. If we let

$$M = \begin{pmatrix}\psi(A^2) & \frac{1}{2}\psi(i[A,B])\\ \frac{1}{2}\psi(i[A,B]) & \psi(B^2)\end{pmatrix} \qquad \text{and} \qquad v = \begin{pmatrix}\alpha\\ \beta\end{pmatrix},$$

we can rewrite this inequality as

$$v^\mathsf{T} M v \geq 0,$$

which simply says that M is positive. Hence,

$$\det M = \psi(A^2)\psi(B^2) - \frac{1}{4}\psi(i[A,B]) \geq 0$$

and thus

$$\sigma_\psi(A)\sigma_\psi(B) \geq \frac{1}{2}|\psi([A,B])|.$$

Consequently, we can already derive Equation (3.4) if $[x, p] = \gamma\hbar\mathbf{1}$ for some $\gamma \in \mathbb{C}$ of norm 1, and in particular the C*-algebra of operators is non-commutative. This suggests to only drop the commutativity constraint from our formalism of classical mechanics, and define quantum observables and quantum states as follows.

Definition 3.72 (Quantum Observables)**.** The observables of a *quantum system* are the self-adjoint elements of a non-commutative C*-algebra A.

Definition 3.73 (Quantum State)**.** The states of a *quantum system* are the normalised positive linear functionals on A.

Indeed, it turns out that these definitions are "correct" in the sense that they are equivalent to the perhaps more well-known Dirac-von Neumann axioms of quantum mechanics, which are phrased in terms of Hilbert spaces. This essentially follows from the Gelfand-Naimark theorem mentioned in Example 3.10. For a more detailed treatment of these ideas refer to [12]. In particular, we see that we obtain quantum groups from classical groups in the same way we obtain quantum mechanics from classical mechanics.

4 Characterising Quantum Isomorphism

> The end of all our exploring will be to arrive where we started and know the place for the first time.
>
> *T.S. Eliot*

Equipped with our knowledge of quantum groups, we now turn to understanding Mančinska and Roberson's homomorphism indistinguishability characterisation of *quantum isomorphism*. Quantum isomorphism is a relation that arises naturally by relaxing definitions of classical isomorphism in a "quantum way".

One such definition is through non-local games: In the *graph isomorphism game*, two players—Alice and Bob—attempt to convice a referee that two graphs G and H are isomorphic. The referee respectively sends a vertex of either G or H to the players. Without communicating, Alice and Bob then have to respond with a vertex of the respective other graph. The players win if the four involved vertices are compatible in a certain sense. It can be shown that Alice and Bob win the game with probability 1 if and only if G and H are isomorphic. We obtain the notion of quantum isomorphism by allowing Alice and Bob to perform measurements on a shared entangled quantum state to determine their responses. This does not allow them to communicate, but allows them to coordinate their responses in ways that are not achievable classically. Analogous to the original game, we say that G and H are quantum isomorphic if Alice and Bob have a strategy to win the game with probability 1.

Another approach, and the one we will take, is to consider graph isomorphism in terms of permutation matrices. Namely, two graphs G and H are isomorphic if and only if there exists a permutation matrix P such that $PA_G = A_H P$. We obtain a notion of quantum isomorphism by instead requiring that $\mathcal{U}A_G = A_H\mathcal{U}$ for a quantum permutation matrix $\mathcal{U}$. That we in fact obtain the same notion of quantum isomorphism as through non-local games has been shown in [19].

In light of these definitions, it is surprising that quantum isomorphism turns out to be equivalent to homomorphism indistinguishability over the class of planar graphs. This chapter is dedicated to retracing the proof of this result—to understand how the seemingly unrelated ideas of quantum correlations and being embeddable in the plane are connected. We will uncover a deep connection between quantum groups and graph homomorphisms through the realm of partition categories and intertwiner spaces.

The story begins with a classical connection between graph isomorphism and the automorphism group of a suitably chosen graph, which can be extended to the quantum case. Specifically, to determine whether graphs G and H are isomorphic, it suffices to know the automorphisms of $G \cup H$. It turns out that an analogous result holds for quantum isomorphism and the *quantum automorphism group*. We will explore this in Section 4.1. We will then see in Section 4.2 that the intertwiners of the quantum automorphism group, and thus the group itself, can be characterised entirely combinatorically in terms of matrices that count homomorphisms from a finitely generated set of graphs. In Section 4.3, we will eventually prove that this set are precisely the planar graphs. Finally, in Section 4.4 we will combine these results into a complete picture and prove that quantum isomorphism is equivalent to homomorphism indistinguishability over planar graphs.

4.1 Quantum and Classical Graph Isomorphism

Let us begin by formalising some key concepts of quantum and classical graph isomorphism. Recall that we can uniquely represent a graph G as an adjacency matrix $A_G \in \{0,1\}^{V(G)\times V(G)}$. One way of characterising graph isomorphism is the following.

Lemma 4.1. *Let G and H be graphs. Then $G \cong H$ if and only if there exists a permutation matrix P such that $A_G P = PA_H$.*

Proof. A permutation is just a bijective map. In particular, for an isomorphism $f\colon V(G) \to V(H), g_i \mapsto h_{f(i)}$, we can define the corresponding permutation matrix as $P_{ij} = [f(i) = j]$. Then $(A_G)_{ij} = (A_H)_{f(i)f(j)} = \sum_{kk'} P_{ik}P_{jk'}(A_H)_{kk'} = (PA_HP^\mathsf{T})_{ij}$ and thus $A_GP = PA_H$. Similarly, if P is a permutation matrix with $A_GP = PA_H$, we define f as the unique map such that $P_{if(i)} = 1$. Then for all i, j we have $g_i \sim g_j \Leftrightarrow (A_G)_{ij} = 1 \Leftrightarrow (PA_HP^\mathsf{T})_{ij} = 1 \Leftrightarrow (A_H)_{f(i)f(j)} = 1 \Leftrightarrow h_{f(i)} \sim h_{f(j)}$. □

In the spirit of this connection, we define quantum isomorphism by moving from permutation matrices to *quantum* permution matrices. Recall that a quantum permutation matrix is a matrix $\mathcal{U} = (u_{ij})$ whose entries are elements of a C*-algebra that satisfy

$$u_{ij}^2 = u_{ij}^* = u_{ij} \quad \text{and} \quad \sum_k u_{ik} = \sum_k u_{kj} = \mathbf{1}.$$

Definition 4.2. We say that two graphs G and H are *quantum isomorphic*, in symbols $G \cong_q H$, if there exists a quantum permutation matrix $\mathcal{U}$ such that $\mathcal{U}A_G = A_H\mathcal{U}$.

In the classical case, there is a straightforward connection between iso- and automorphisms. In particular, to determine whether two graphs G and H are isomorphic, it suffices to know the automorphisms of $G \cup H$.

Lemma 4.3. *Let G and H be connected graphs. Then $G \cong H$ if and only if there exist $g \in V(G)$ and $h \in V(H)$ that are in the same orbit of* $\mathrm{Aut}(G \cup H)$.

Proof. First note that an isomorphism $f\colon G \to H$ induces an automorphism of $G \cup H$ that maps $V(G)$ to $f(V(G)) = V(H)$ and $V(H)$ to $f^{-1}(V(H)) = V(G)$. In particular, any $g \in V(G)$ is in the same orbit as its image $f(g) \in V(H)$. On the other hand, if $g \in V(G)$ and $h \in V(H)$ are in the same orbit, there exists an automorphism f with $f(g) = h$. Since G and H are connected, we must have $f(V(G)) = V(H)$ and thus $f|_{V(G)}$ is an isomorphism $G \to H$. □

It turns out that an analogous result holds in the quantum case. However, it is not a priori clear how to define a quantum analogue of the automorphism group of a graph. We will proceed similar to our definition of quantum isomorphism. Observe that a classical automorphism of a graph G corresponds, in the sense of Lemma 4.1, to a permutation matrix P such that $PA_G = A_GP$. Replacing P by a quantum permutation matrix $\mathcal{U}$ as in Definition 4.2 yields $\mathcal{U}A_G = A_G\mathcal{U}$, which is just saying that A_G is a $(1,1)$-intertwiner of $\mathcal{U}$.

Recall that by Theorem 3.59, compact matrix quantum groups are completely determined by their intertwiners. The idea is thus to define the *quantum automorphism group* as the compact matrix subgroup of S_n^+ that also has A_G as an intertwiner. Since we already know that the intertwiners of S_n^+ are generated by the multiplication maps $M^{1,0}$ and $M^{1,2}$, we can state this formally as follows.

Definition 4.4. The *quantum automorphism group* $\mathrm{Qut}(G)$ of a graph G is the unique compact matrix quantum group whose intertwiners are given by $\langle M^{1,0}, M^{1,2}, A_G\rangle_{+,\circ,\otimes,*}$.

We introduced orbits of compact matrix quantum groups in Section 3.3.1, and their special form in the case of quantum symmetric groups in Theorem 3.50. We can thus state the following quantum analogue of Lemma 4.3 which has been proved in [19].

Lemma 4.5. *Let G and H be connected graphs. Then $G \cong_q H$ if and only if there exist $g \in V(G)$ and $h \in V(H)$ that are in the same orbit of* $\mathrm{Qut}(G \cup H)$.

Since the quantum automorphism group of a graph G is not really a tangible object, we will mainly reason through its intertwiners. Therefore, the following alternative characterisation of the orbits of $\mathrm{Qut}(G)$ will be useful. Here we call a vector $w \in \mathbb{C}^{V(G)}$ *constant on the orbits of* $\mathrm{Qut}(G)$ if $w_u = w_v$ whenever u, v are in the same orbit.

Lemma 4.6. *Let G be a graph and $\mathcal{U}$ the fundamental representation of* $\mathrm{Qut}(G)$. *Then a vector $w \in \mathbb{C}^{V(G)}$ is constant on the orbits of* $\mathrm{Qut}(G)$ *if and only if w is a $(1,0)$-intertwiner, that is, $\mathcal{U}w = w(\mathbf{1})$.*

Proof. Recall that two vertices v_i, v_j are defined to be in the same orbit of $\mathrm{Qut}(G)$, written $v_i \sim v_j$, if $u_{ij} \neq 0$. If w is constant on the orbits of $\mathrm{Qut}(G)$ then we have $(\mathcal{U}w)_i = \sum_k u_{ik} w_k = \sum_{k \sim i} u_{ik} w_k = w_i \sum_k u_{ik} = w_i \mathbf{1} = (w\mathbf{1})_i$. On the other hand, for i, j with $v_i \sim v_j$ we have $(\mathcal{U}w)_i = \sum_k u_{ik} w_k = w_i \mathbf{1}$. One can show that in any C*-algebra A, projectors $p_1, \dots, p_n \in A$ with $\sum_i p_i = \mathbf{1}$ satisfy $p_i p_j = \delta_{ij} p_i$. Multiplying by $u_{i\ell}$ for some fixed ℓ on both sides therefore yields $u_{i\ell} w_\ell = u_{i\ell} w_i$. This in turn is satisfied if and only if $w_i = w_\ell$ whenever $u_{i\ell} \neq 0$, that is, if w is constant on the orbits of $\mathrm{Qut}(G)$. □

For example, the lemma implies that two vertices g, h are in the same orbit of $\mathrm{Qut}(G \cup H)$ if and only if $w_g = w_h$ for all $(1,0)$-intertwiners w. Given these characterisations of quantum isomorphism, let us now explore how intertwiners and homomorphism counts are related.

4.2 Intertwiners and Homomorphism Tensors

The central insight is that the intertwiners of compact matrix quantum groups can be characterised combinatorically through matrices whose entries count homomorphisms depending on the position of certain labels in the target graph. These homomorphism tensors have already proven useful in other places, such as in [15], where they were used to derive a unified framework for homomorphism indistinguishability characterisations of certain algebraic equation systems. To define homomorphism tensors, we first have to introduce bilabelled graphs.

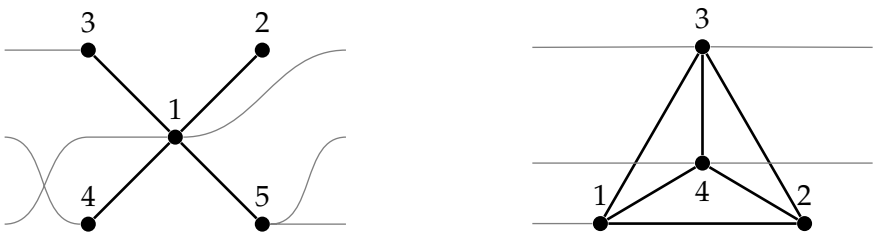
4.2.1 Bilabelled Graphs

As the name suggests, bilabelled graphs are graphs whose vertices may be annotated by two types of labels, the *in-labels* and the *out-labels*. A vertex can have both in-labels and out-labels and not every vertex has to be labelled. We formalise this as follows.

Definition 4.7. A (k, ℓ)*-bilabelled graph* $\mathbf{K}$ is a tuple $(K, \mathbf{a}, \mathbf{b})$ where K is a graph, $\mathbf{a} = (a_1, \dots, a_k) \in V(K)^k$ is the *output vector*, and $\mathbf{b} = (b_1, \dots, b_\ell) \in V(K)^\ell$ is the *input vector*. We let $V(\mathbf{K}) = V(K)$ and $E(\mathbf{K}) = E(K)$. We denote the set of all (k, ℓ)-bilabelled graphs by $\mathcal{G}(k, \ell)$.

The definition of labels in terms of input- and output vectors is often convenient. However, we can equivalently think of the labels as elements in $[k]$ and $[\ell]$ respectively, where a vertex $v \in V(K)$ has out-label $i \in [k]$ if $a_i = v$, and in-label $j \in [\ell]$ if $b_j = v$. In this sense, we also denote by $l_{in}(v) \subseteq [\ell]$, $l_{out}(v) \subseteq [k]$, and $l(v) \subseteq [\ell] \sqcup [k]$ the sets of in-labels, out-labels, and all labels of v respectively. Graphically, we draw labels as grey "wires" that are ordered from top to bottom, where input wires extend to the right and output wires to the left. For an example, see Figure 4.1. We will see in a bit why this notation is useful.

On the basis of the labels, we can define operations on bilabelled graphs, similar to how we did for partitions. Let $\mathbf{K} = (K, \mathbf{a}, \mathbf{b}) \in \mathcal{G}(\ell, k)$ and $\mathbf{K}' = (K', \mathbf{a}', \mathbf{b}') \in \mathcal{G}(\ell', k')$. Then the *tensor product* $\mathbf{K} \otimes \mathbf{K}' \in \mathcal{G}(\ell + \ell', k + k')$ is defined as the bilabelled graph $(K \cup K', \mathbf{a}\mathbf{a}', \mathbf{b}\mathbf{b}')$. If $\ell' = k$, the *composition* $\mathbf{K}' \circ \mathbf{K} \in \mathcal{G}(k', \ell)$ is formed by constructing the disjoint union $K \cup K'$, identifying the vertices a'_i and b_i for all $i \in [k]$, and removing any duplicate edges but keeping the loops.



$\mathbf{K} = (K_{1,4}, (3,4,1), (1,5,5))$ $\mathbf{K}' = (K_4, (3,4,1), (3,4))$

Figure 4.1: Drawing bilabelled graphs

Finally, the *adjoint* $\mathbf{K}^* \in \mathcal{G}(k,\ell)$ is given by $(K, \mathbf{b}, \mathbf{a})$. We denote the closure of a set of bilabelled graphs $\mathbf{K}_1, \ldots, \mathbf{K}_n$ under these operations by $\langle \mathbf{K}_1, \ldots, \mathbf{K}_n \rangle_{\circ,\otimes,*}$. As for partitions, we can view the operations graphically, as shown in Figure 4.2. Composition, in particular, corresponds to drawing $\mathbf{K}'$ on the left of $\mathbf{K}$ so that the output and input wires align, connecting the wires, and then contracting the resulting edges.

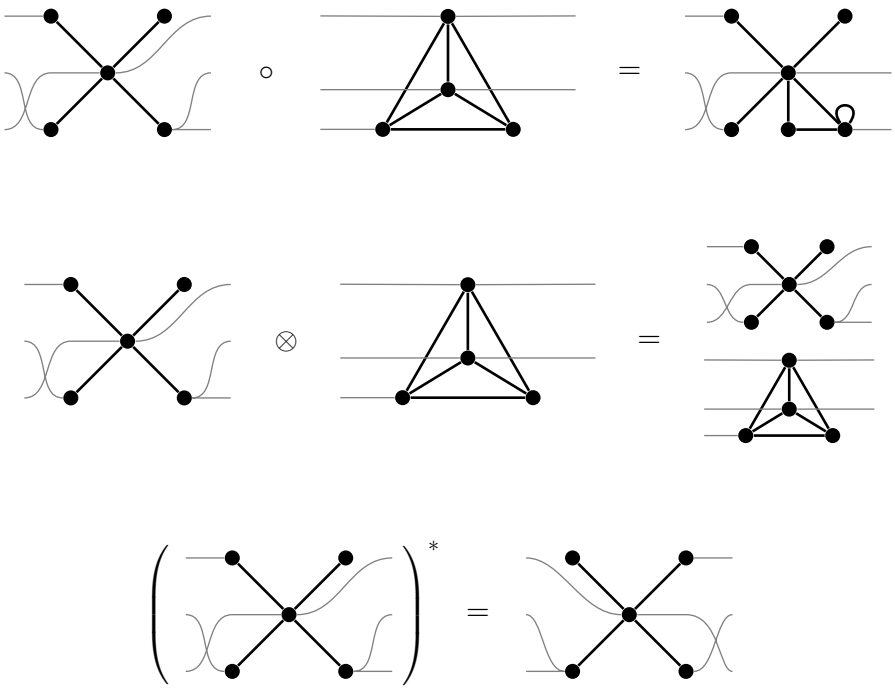


Figure 4.2: Operations on bilabelled graphs

It turns out that these operations are more than just similar to those on partitions. Consider some partition $P = \{P_1, \ldots, P_n\} \in \mathbb{P}(k,\ell)$. We construct a bilabelled graph $\mathbf{P} = \Gamma(P)$ by setting $V(\mathbf{P}) = \{v_i \mid i \in [n]\}$ and $E(\mathbf{P}) = \varnothing$, as well as $l_{in}(v_i) = \{i \mid i \in [\ell], i_U \in P_i\}$ and $l_{out}(v_i) = \{i \mid i \in [k], i_L \in P_i\}$. Then it is a routine calculation to verify that $\Gamma(P \otimes P') = \Gamma(P) \otimes \Gamma(P')$, $\Gamma(P \circ P') = \Gamma(P) \circ \Gamma(P')$, and $\Gamma(P^*) = \Gamma(P)^*$. Furthermore, Γ has an inverse that assigns to an edgeless bilabelled graph $\mathbf{K} = (K, \mathbf{a}, \mathbf{b})$ the partition $\mathbb{P}_{\mathbf{K}} = \Gamma^{-1}(\mathbf{K}) = \{\{i_U \mid i \in l_{in}(v)\} \cup \{i_L \mid i \in l_{out}(v)\} \mid v \in V(K)\}$.

Proposition 4.8. *There exists a bijection between partitions and edgeless bilabelled graphs that commutes with tensor product, composition, and taking adjoints.*

Some important bilabelled graphs that correspond to partitions are the generalised multiplication graphs $\mathbf{M}^{k,\ell} = \Gamma(M^{k,\ell})$ and the swap graph $\mathbf{S} = \Gamma(S)$.

4.2.2 Counting Bilabelled Graph Homomorphisms

It turns out that just as for partitions, we can associate linear maps to bilabelled graphs. Their matrix representations are given by *homomorphism tensors*.

Definition 4.9. A *bilabelled graph homomorphism* from $\mathbf{K} = (K, \mathbf{a}, \mathbf{b}) \in \mathcal{G}(k, \ell)$ to $\mathbf{K}' = (K', \mathbf{a}', \mathbf{b}') \in \mathcal{G}(k, \ell)$ is a homomorphism $h\colon K \to K'$ such that $h(\mathbf{a}) = \mathbf{a}'$ and $h(\mathbf{b}) = \mathbf{b}'$.

Definition 4.10. Given a bilabelled graph $\mathbf{K} \in \mathcal{G}(k, \ell)$ and a graph G, the *G-homomorphism tensor* of $\mathbf{K}$ is the $V(G)^k \times V(G)^\ell$ matrix $\mathbf{K}_G$ with entries

$$(\mathbf{K}_G)_{\mathbf{uv}} = \#\,\text{bilabelled graph homomorphisms } h\colon \mathbf{K} \to (G, \mathbf{u}, \mathbf{v}).$$

Remark 4.11. Technically, for $\mathbf{K}_G$ to define a unique linear map $(\mathbb{C}^n)^{\otimes \ell} \to (\mathbb{C}^n)^{\otimes k}$ we need to fix an ordering on $V(G)^\ell$ and $V(G)^k$. We will always assume a lexicographic ordering with respect to some ordering of $V(G)$.

Let us discuss some important examples of homomorphism tensors.

Example 4.12. If $\mathbf{K} \in \mathcal{G}(0, 0)$, then $\mathbf{K}_G$ is a 1×1 matrix, which we may interpret as a scalar, whose single entry is $\hom(K, G)$. ⌟

Example 4.13. Consider the edgeless bilabelled graphs $\mathbf{M}^{\ell,k}$ associated to the single-block partitions $\{\{1_L, \dots, \ell_L, 1_U, \dots, k_U\}\}$. Then $\mathbf{M}^{\ell,k}$ is a single-vertex graph with k in-labels and ℓ out-labels. Since $\mathbf{M}^{\ell,k}$ has no edges, $\mathbf{M}_G^{\ell,k}$ depends only on $|V(G)| =: n$ and we have

$$(\mathbf{M}_G^{\ell,k})_{\mathbf{uv}} = \begin{cases} 1 & \text{if } u_1 = \cdots = u_\ell = v_1 = \cdots = v_k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have $\mathbf{M}_G^{\ell,k} = M^{(n),\ell,k}$ for all $\ell, k \in \mathbb{N}$. An important special case is $\mathbf{M}_G^{1,1}$, which is just the identity matrix I_n. We thus also write $\mathbf{I}$ for $\mathbf{M}^{1,1}$. ⌟

Example 4.14. Consider the *bilabelled edge* $\mathbf{A} = (K_2, (v_1), (v_2))$. Then for any graph G

$$(\mathbf{A}_G)_{vw} = \begin{cases} 1 & \text{if } vw \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mathbf{A}_G$ is just the adjacency matrix A_G of G. ⌟

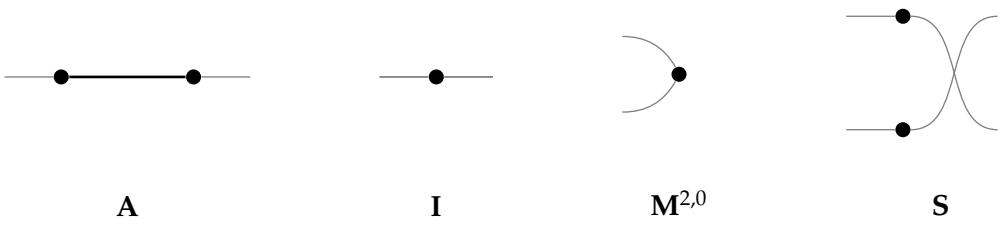


Figure 4.3: Some important bilabelled graphs.

Mančinska and Roberson now showed that Proposition 3.62—Banica and Speichers result on the compatibility of operations on partitions and on the corresponding linear maps—can be generalised to bilabelled graphs and homomorphism tensors.

Lemma 4.15. *Let* $\mathbf{K}, \mathbf{K}' \in \boldsymbol{\mathcal{G}}$. *Then we have* $(\mathbf{K} \otimes \mathbf{K}')_G = \mathbf{K}_G \otimes \mathbf{K}'_G$, $(\mathbf{K}^*)_G = \mathbf{K}_G^*$, *and if* $\mathbf{K}$ *and* $\mathbf{K}'$ *are compatible,* $(\mathbf{K} \circ \mathbf{K}')_G = \mathbf{K}_G \circ \mathbf{K}'_G$.

The proof is essentially just a tedious case-by-case verification, so we omit it here. In particular, this allows us to define *graph categories* which correspond to concrete tensor categories with duals, just as for partition categories.

Definition 4.16. A *graph category* $\mathcal{F}$ is a collection of bilabelled graphs, such that

(i) if $\mathbf{K} \in \mathcal{F}(k, \ell)$ and $\mathbf{K}' \in \mathcal{F}(r, s)$, then $\mathbf{K} \otimes \mathbf{K}' \in \mathcal{F}(k + r, \ell + s)$,

(ii) if $\mathbf{K} \in \mathcal{F}(k, \ell)$ and $\mathbf{K}' \in \mathcal{F}(\ell, r)$, then $\mathbf{K} \circ \mathbf{K}' \in \mathcal{F}(k, r)$,

(iii) if $\mathbf{K} \in \mathcal{F}(k, \ell)$ then $\mathbf{K}^* \in \mathcal{F}(\ell, k)$,

(iv) $\mathbf{I} \in \mathcal{F}(1, 1)$,

(v) $\mathbf{M}^{2,0} \in \mathcal{F}(2, 0)$,

where $\mathcal{F}(k, \ell) = \mathcal{F} \cap \boldsymbol{\mathcal{G}}(k, \ell)$.

The previous lemma thus implies that a graph category induces a concrete tensor category with duals C for every graph G by letting

$$C(\ell, k) = \operatorname{span}\{\mathbf{K}_G \mid G \in \mathcal{F}\}.$$

This also means that every pair of a graph category $\mathcal{F}$ and a graph G corresponds to a unique compact matrix quantum group. Mančinska and Roberson call these quantum groups *graph-theoretical*.

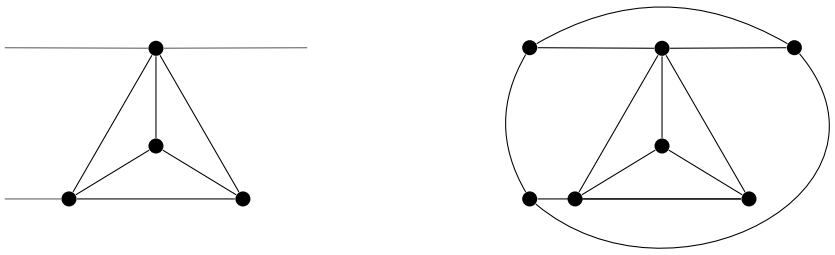
Observation 4.17. *By Definition 4.4, the quantum automorphism group of a graph* G *is exactly the graph-theoretical quantum group corresponding to* $\langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A} \rangle_{\circ,\otimes,*}$, *that is,*

$$C^{\mathrm{Qut}(G)} = \operatorname{span}\{\mathbf{K}_G \mid \mathbf{K} \in \langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A} \rangle_{\circ,\otimes,*}\}.$$

Remark 4.18. While we do not allow multiple edges, loops arise very naturally in this formalism and removing them breaks the correspondence between the operations on graphs and on homomorphism tensors. This remains the case even if the target graph is loopless. However, when we later talk about homomorphism indistinguishability we may restrict ourselves to simple graphs again. This is because for simple graphs G, H, we have $\hom(F, G) = 0 = \hom(F, H)$ for all graphs F that contain a loop.

4.3 Constructing Planar Graphs

We have seen that we can characterise the quantum automorphism group of a graph G through the graph category $\langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A} \rangle_{\circ,\otimes,*}$. The natural question to ask is what this graph category looks like, that is, which graphs it contains. It turns out that $\langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A} \rangle_{\circ,\otimes,*}$ are precisely the *planar bilabelled graphs* $\boldsymbol{\mathcal{P}}$. In this section, we want to outline Mančinska and Roberson's proof of this result. We omit any technical details and instead focus on intuitively understanding the results.



K K°

Figure 4.4: A planar bilabelled graph.

4.3.1 Planar Bilabelled Graphs

We first need to define a notion of planarity for bilabelled graphs. The idea is that a planar embedding of a bilabelled graph should also respect the input and output wires. To that end, we introduce the *enveloping cycle*.

Definition 4.19. For a bilabelled graph $\mathbf{K} = (K, \mathbf{a}, \mathbf{b}) \in \mathcal{G}(k, \ell)$ we define $K^\circ := K^\circ(\mathbf{a}, \mathbf{b})$ as the graph obtained from K by adding the cycle $C = \alpha_1, \dots, \alpha_k, \beta_\ell, \dots, \beta_1$ and adding the edges $a_i\alpha_i$ and $b_j\beta_j$ for all $i \in [k]$ and $j \in [\ell]$. We refer to C as the *enveloping cycle*[1] of K°. Moreover, we define $K^\odot := K^\odot(\mathbf{a}, \mathbf{b})$ by adding an additional vertex adjacent to all vertices on the enveloping cycle.

Although K° and $K^\odot$ depend on the input and output vectors of $\mathbf{K}$, they will usually be clear from the context, so we will omit them in our notation. Using these graphs, we can define planar bilabelled graphs.

Definition 4.20. For any $k, \ell \in \mathbb{N}$, the planar (k, ℓ)-bilabelled graphs are given by

$$\mathcal{P}(k, \ell) = \{\mathbf{K} \in \mathcal{G}(k, \ell) \mid K^\odot \text{ is planar}\},$$

and we let $\mathcal{P} = \bigcup_{k,\ell \in \mathbb{N}} \mathcal{P}(k, \ell)$. In particular, $\mathcal{P}(0, 0)$ are just the planar graphs.

One can show that a cycle in a planar graph is facial if and only if the graph obtained by adding a new vertex adjacent to each vertex of the cycle is still planar. Hence, an alternative definition of planar bilabelled graphs is the following.

Definition 4.21. For any $k, \ell \in \mathbb{N}$, the planar (k, ℓ)-bilabelled graphs are given by

$$\mathcal{P}(k, \ell) = \{\mathbf{K} \in \mathcal{G}(k, \ell) \mid K^\circ \text{ is planar and its enveloping cycle is facial}\}.$$

We first want to show that all graphs in $\langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A} \rangle_{\circ,\otimes,*}$ are planar bilabelled graphs. To that end, we state a few results on operations that preserve planarity. The proofs are straightforward.

Lemma 4.22. *Let G be a planar graph and $u, v \in V(G)$. If G' is a graph obtained from G by either removing u, removing uv, subdividing uv, or contracting uv, then G' is planar.*

Lemma 4.23. *Let G be a planar graph, $u, v, w \in V(G)$, and $uv, vw \in E(G)$. Let G' be the graph obtained by subdividing the edge vw with vertex w_1, and adding the edge uw_1. Then G' is planar.*

[1] In the case $k + \ell = 2$, Mančinska and Roberson take the enveloping cycle to consist of two vertices connected by two edges, hence K° is a multigraph. This is done to avoid special cases in their proofs and does not change the definition of $\mathcal{P}$.

Lemma 4.24 (Lemma 4.11 in [22]). *Let G be a planar graph and $v \in V(G)$, such that there exists a planar embedding in which the edges incident to v occur in cyclic order $e_1, \dots, e_k$. Let G' be the graph obtained by subdividing each e_i with vertices v_i and adding the edges $v_k v_1$ and $v_i v_{i+1}$ for $i < k$. Then G' is planar.*

Lemma 4.25 (Lemma 4.12 in [22]). *Let G and H be planar graphs with facial cycles $C = u_1, \dots, u_k$ and $D = v_1, \dots, v_\ell$ respectively. Let X be the graph obtained from $G \cup H$ by adding the edges $u_1 v_1, \dots u_m v_m$ for some $m \leq \min(k, \ell)$. Then X is planar and $u_m, u_{m+1}, \dots, u_\ell, u_1, v_1, v_k, v_{k-1}, \dots, v_m$ is a facial cycle.*

Using these lemmas, one can prove that composition, tensor product, and taking adjoints preserve planarity by reducing them to the listed operations. Since $\mathbf{M}^{1,0}, \mathbf{M}^{1,2}$, and $\mathbf{A}$ are obviously planar, we get the following result.

Lemma 4.26. $\langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A} \rangle_{\circ,\otimes,*} \subseteq \mathcal{P}$.

4.3.2 Constructing Planar Bilabelled Graphs

It remains to show that we can in fact construct any planar bilabelled graph. For that we use the following collection of basic building blocks. For simplicity, we only construct loopless graphs. The general case is essentially analogous, using slightly modified versions of these base graphs.

Definition 4.27. Let S^d be the star graph on $d+1$ vertices with vertex set $\{v, v_1, \dots, v_d\}$ and edges $\{v v_i \mid i \in [d]\}$. For all $m, d \in \mathbb{N}$, we define $\mathbf{S}^{m,d} := (S^d, (v^m), (v_1, \dots, v_d))$, $\mathbf{S}_L^{m,d} := (S^d, (v^m), (v, v_1, \dots, v_d))$, and $\mathbf{S}_R^{m,d} := (S^d, (v^m), (v_1, \dots, v_d, v))$.

Importantly, these graphs can all be constructed from $\mathbf{M}^{1,2}$, $\mathbf{M}^{2,0}$, and $\mathbf{A}$.

Lemma 4.28 (Lemma 5.18 in [22]). *For all $m, d \in \mathbb{N}$, we have $\mathbf{S}^{m,d}, \mathbf{S}_L^{m,d}, \mathbf{S}_R^{m,d} \in \langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A} \rangle_{\circ,\otimes,*}$.*

We first note a necessary condition regarding the labels of a planar bilabelled graph. Recall the bijection Γ between partitions and edgeless bilabelled graphs. We can also associate a partition to an arbitrary bilabelled graph by simply ignoring the edges. A bit of thought then shows that any partition associated to a planar bilabelled graph must be non-crossing.

Lemma 4.29 (Lemma 6.3 in [22]). *For any bilabelled graph $\mathbf{K} \in \mathcal{P}$, the partition $\mathbb{P}_{\mathbf{K}}$ is non-crossing.*

Central to our construction of planar bilabelled graphs will be the following technical statements about non-crossing partitions. To state them concisely we consider indices modulo n.

Lemma 4.30 (Lemma 6.4 in [22]). *Let V be a set and let $\mathbf{c} = (c_1, \dots, c_n) \in V^n$ such that $\mathbb{P}_{\mathbf{c}}$ is non-crossing. If there exists a $v \in V$ and $i, r \in [n]$ such that $c_i = v = c_{i+r}$ and $c_{i+s} \neq v$ for all $s \in [r-1]$, then there exists a $u \in V$ that occurs consecutively in $\mathbf{c}$ and only occurs among $c_{i+1}, \dots, c_{i+r-1}$.*

Corollary 4.31 (Corollary 6.5 in [22]). *Let V be a set and $\mathbf{c} \in V^n$ such that $\mathbb{P}_{\mathbf{c}}$ is non-crossing. Then there exists a $v \in V$ that occurs consecutively in $\mathbf{c}$. In particular, if $\mathbb{P}_{\mathbf{c}} = \mathbb{P}_{\mathbf{K}}$ for some $\mathbf{K} = (K, \mathbf{a}, \mathbf{b}) \in \mathcal{P}(\ell, k)$, then there exists a vertex $v \in V(K)$ that occurs consecutively in $(a_1, \dots, a_\ell, b_k, \dots, b_1)$.*

In particular, these results imply that for any planar bilabelled graph, there must exist a vertex whose labels are not "interleaved" with the remaining labels. The idea is that we can add such a vertex to a planar bilabelled graph using the basic graphs introduced in Definition 4.27. This allows us to prove the following lemma. Since we will later adapt this construction when investigating other graph categories, we include a proof.

Lemma 4.32 (Lemma 6.6 in [22]). *Let $\mathbf{K} \in \mathcal{P}$ with $V(\mathbf{K}) \geq 2$. Then there exists a bilabelled graph $\mathbf{K}' \in \mathcal{P}$ with $|V(\mathbf{K}')| = |V(\mathbf{K})| - 1$ and $m, d, r, t \in \mathbb{N}$ such that one of the following holds:*

$$\mathbf{K} = (\mathbf{I}^{\otimes r} \otimes \mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes t}) \circ \mathbf{K}' \tag{4.1}$$

$$\mathbf{K} = \mathbf{K}' \circ (\mathbf{I}^{\otimes r} \otimes \mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes t})^* \tag{4.2}$$

$$\mathbf{K} = (\mathbf{S}_L^{m,d} \otimes \mathbf{I}^{\otimes t}) \circ (\mathbf{M}^{1,r} \otimes \mathbf{K}') \tag{4.3}$$

$$\mathbf{K} = (\mathbf{I}^{\otimes t} \otimes \mathbf{S}_R^{m,d}) \circ (\mathbf{K}' \otimes \mathbf{M}^{1,r}) \tag{4.4}$$

Proof. Suppose first that $\mathbf{K} = (K, \mathbf{a}, \mathbf{b}) \in \mathcal{P}(\ell, k)$ with $\ell + k \geq 1$. By Corollary 4.31, there exists a $v \in V(\mathbf{K})$ that occurs consecutively in $(a_1, \ldots, a_\ell, b_k, \ldots, b_1)$. We make the following case distinction.

Case 1: *v occurs only in* $\mathbf{a}$. In this case v might occur consecutively by appearing in $a_1, \ldots, a_p$ and $a_q, \ldots, q_\ell$ for some $p < q \in [\ell]$. However, if this is the case then we can find, by Corollary 4.31, a $v' \in V(\mathbf{K})$ that occurs consecutively and only among $a_{p+1}, \ldots, a_{q-1}$. Hence, we may assume that the occurences of v are precisely $a_p, \ldots, a_q$ for some $p, q \in [\ell]$. Now let $v_1, \ldots, v_d$ be the neighbours of v, and set $\mathbf{K}' = (K', \mathbf{a}', \mathbf{b})$, where $K' = K - v$ and $\mathbf{a}' = (a_1, \ldots, a_{p-1}, v_1, \ldots, v_d, a_{q+1}, \ldots, a_\ell)$. Then

$$\mathbf{K} = (\mathbf{I}^{\otimes r} \otimes \mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes t}) \circ \mathbf{K}',$$

where $r = p - 1$, $m = q - p + 1$, and $t = \ell - q$. It remains to show that $\mathbf{K}'$ is planar. This is proven by constructing K'° from K° using the operations listed in Lemmas 4.22 and 4.24. For brevity, we show this procedure graphically in Figure 4.5.

Case 2: *v occurs only in* $\mathbf{b}$. In this case, $\mathbf{K}^*$ falls into Case 1. Consequently, there exists a planar bilabelled graph $\mathbf{K}'$ such that

$$\mathbf{K}^* = (\mathbf{I}^{\otimes r} \otimes \mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes t}) \circ \mathbf{K}'.$$

Taking the adjoint of both sides yields

$$\mathbf{K} = \mathbf{K}'^* \circ (\mathbf{I}^{\otimes r} \otimes \mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes t})^*,$$

and since taking adjoints preserve planarity, we have $\mathbf{K}'^* \in \mathcal{P}$.

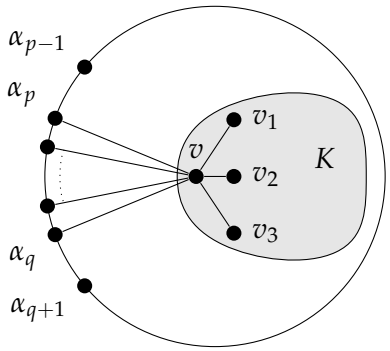
Case 3: $a_1 = b_1 = v$. Note that we can assume that the occurences of v are precisely $a_1, \ldots, a_m$ and $b_1, \ldots, b_r$. Otherwise, it must be that all entries of either $\mathbf{a}$ or $\mathbf{b}$ are equal to v. Hence by Corollary 4.31, there exists a $v' \in V(\mathbf{K}$ that only occurs among $\mathbf{a}$ or $\mathbf{b}$, and thus Case 1 or Case 2 apply. Now let $v_1, \ldots, v_d$ be the neighbours of v in $\mathbf{K}$ and set $\mathbf{K}' = (K', \mathbf{a}', \mathbf{b}')$, where $K' = K - v$, $\mathbf{a}' = (v_1, \ldots, v_d, a_{m+1}, \ldots, a_\ell)$, and $\mathbf{b}' = (b_{r+1}, \ldots, b_k)$. Let $t = \ell - m$. Then we have

$$\mathbf{K} = (\mathbf{S}_L^{m,d} \otimes \mathbf{I}^{\otimes t}) \circ (\mathbf{M}^{1,r} \otimes \mathbf{K}').$$

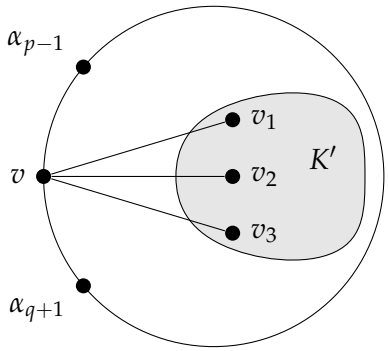
To show that $\mathbf{K}' \in \mathcal{P}$, we reduce to Case 1. Consider $\hat{\mathbf{K}} = \mathbf{K} \circ (\mathbf{M}^{r,0} \otimes \mathbf{I}^{\otimes(k-r)})$. This removes the occurences of v from $\mathbf{b}$, so it falls into Case 1. Moreover, the $\hat{\mathbf{K}}'$ constructed using the approach of Case 1 is exactly the $\mathbf{K}'$ constructed here. In particular, $\mathbf{K}' \in \mathcal{P}$.

Case 4: $a_\ell = b_k = v$. This case is symmetric to Case 3.

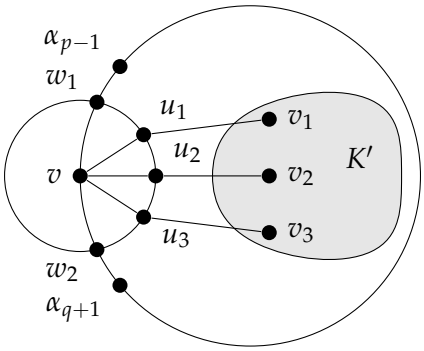
Case 5: $k + \ell = 0$. In this case there is no enveloping cycle. We thus choose an arbitrary vertex $v \in V(\mathbf{K})$ with neighbours $v_1, \ldots, v_d$ and let $\mathbf{K}' = (K', (v_1, \ldots, v_d), ())$, where $K' = K - v$. Then $\mathbf{K} = \mathbf{S}^{0,d} \circ \mathbf{K}'$. To see that $\mathbf{K}'$ is planar, fix some planar embedding of K where $v_1, \ldots v_d$ occur in cyclic order. Subdivide the edges vv_i with vertices α_i and add edges to create a cycle $C = \alpha_1, \ldots, \alpha_d$. By Lemma 4.24, the resulting graph is planar and in fact isomorphic to $K'^{\odot}$. It follows that $\mathbf{K}' \in \mathcal{P}$. □



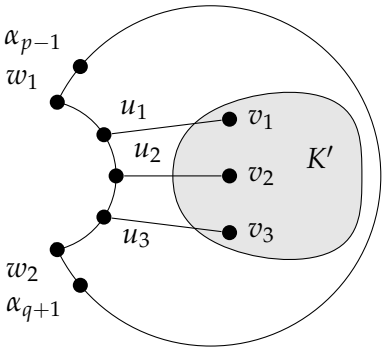
The vertex v and its neighbours in K°.



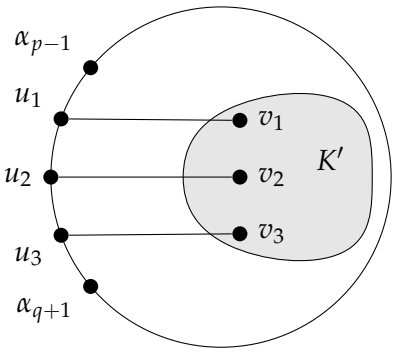
Remove edges $v\alpha_i$ for $i \in \{p+1, \ldots, q\}$, contract $v\alpha_p$, and unsubdivide $v, \ldots, \alpha_{q+1}$.



Subdivide edges incident to v and add cycle through new vertices.



Remove vertex v and edge w_1w_2.



Unsubdivide $\alpha_{p-1}w_1u_1$ and u_3w_2, α_q to obtain K'°.

Figure 4.5: Proving that $\mathbf{K}' \in \boldsymbol{\mathcal{P}}$.

The main result of this section now follows from Lemma 4.32 via an inductive argument.

Theorem 4.33 (Theorem 6.7 in [22]). $\boldsymbol{\mathcal{P}} = \langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A} \rangle_{\circ,\otimes,*}$.

4.4 Quantum Isomorphism is Homomorphism Indistinguishability over Planar Graphs

So far we have seen that a graph category gives rise to a compact matrix quantum group for each graph G, whose intertwiners are given by homomorphism tensors. Furthermore, the graph category corresponding to the quantum automorphism group of a graph turns out to coincide with the class of planar bilabelled graphs. The desired result of this chapter thus seems in close reach. Alas, the remaining argument is still more involved than one might expect. We first need to introduce some additional machinery that allows us to relate intertwiner spaces. Since we will only work with quantum automorphism groups for now, let us write C_q^G for the intertwiners $C^{\mathrm{Qut}(G)}$ of the quantum automorphism group of a graph G.

Definition 4.34. Let G and H be graphs. A *weak isomorphism* between $\mathrm{Qut}(G)$ and $\mathrm{Qut}(H)$ is a bijection $\Phi\colon C_q^G \to C_q^H$ such that for all $T, T' \in C_q^G$,

(i) $\Phi(M^{1,2}) = M^{1,2}, \Phi(M^{1,0}) = M^{1,0}$, and $\Phi(A_G) = A_H$,

(ii) $\Phi(TT') = \Phi(T)\Phi(T')$ if T, T' are compatible,

(iii) $\Phi(T \otimes T') = \Phi(T) \otimes \Phi(T')$,

(iv) $\Phi(T^*) = \Phi(T)^*$,

(v) For all $\ell, k \in \mathbb{N}$, the restriction of Φ to $C_q^G(\ell, k)$ is a bijective linear map to $C_q^H(\ell, k)$.

Remark 4.35. Technically, we would need to distinguish the intertwiners $M^{1,2}$ and $M^{1,0}$ living in C_q^G from those living in C_q^H. However, we will see that the existence of a weak isomorphism between $\mathrm{Qut}(G)$ and $\mathrm{Qut}(H)$ implies that $|V(G)| = |V(H)|$, so we can identify the maps with each other.

Remark 4.36. In the original paper, Mančinska and Roberson do not require that a weak isomorphism Φ satisfies $\Phi(A_G) = A_H$, and instead treat this condition seperately. However, we will later see that it is more natural to include it in the definition.

We now want to prove the following theorem.

Theorem 4.37 (Theorem 7.16 in [22])**.** *Let G and H be graphs. Then the following are equivalent.*

(*i*) *G and H are quantum isomorphic.*

(*ii*) *There exists a weak isomorphism between* $\mathrm{Qut}(G)$ *and* $\mathrm{Qut}(H)$*.*

(*iii*) *G and H are homomorphism indistinguishable over the class of all planar graphs.*

For brevity, if two graphs are homomorphism indistinguishable over the class of planar graphs we also call them *planar isomorphic*. The theorem is proved by showing three implications. We first show that quantum isomorphism of G and H implies weak isomorphism of $\mathrm{Qut}(G)$ and $\mathrm{Qut}(H)$. We then show that the existence of such a weak isomorphism implies that G and H are planar isomorphic. Finally, for the converse, we show that if G and H are planar isomorphic then they are also quantum isomorphic. The last implication is by far the most involved and makes use of the properties of quantum isomorphism that we have derived in the beginning of this chapter. The other two implications are more straightforward, and we will prove generalised versions of them in Chapter 5. To avoid unnecessary repetition while leaving the treatment of general quantum groups as self-contained as possible, we focus here on their main ideas and leave the formal proofs for Chapter 5.

4.4.1 Via Weak Isomorphisms to Planar Isomorphism

Suppose then that G and H are quantum isomorphic. Then per definition there exists a quantum permutation matrix $\mathcal{U}$ such that $\mathcal{U}A_G = A_H\mathcal{U}$. Moreover, a routine calculation shows that $\mathcal{U}M^{1,0} = M^{1,0}\mathcal{U}$ and $\mathcal{U}M^{1,2} = M^{1,2}\mathcal{U}$. Hence, one might hope to define our desired weak isomorphism $\Phi\colon C_q^G \to C_q^H$ through

$$\mathcal{U}^{\otimes k}T = \Phi(T)\mathcal{U}^{\otimes \ell}$$

for all $T \in C_q^G(k,\ell)$. Indeed, the invertibility of $\mathcal{U}$ implies that this is well defined: If $T'\mathcal{U}^{\otimes \ell} = \mathcal{U}^{\otimes k}T = T''\mathcal{U}^{\otimes \ell}$, then $T' = T''$. A similar argument moreover shows that Φ must be injective. However, it is a priori not clear that for every $T \in C_q^G$ there exists a $T' \in C_q^H$ such that $\mathcal{U}^{\otimes k}T = T'\mathcal{U}^{\otimes \ell}$. Now one can verify that Φ is linear and commutes with tensor product, composition, and taking adjoints. Since $C_q^G = \langle M^{1,0}, M^{1,2}, A_G\rangle_{+,\circ,\otimes,*}$, and Φ is defined on these generators, this also implies that Φ is indeed defined everywhere on C_q^G. Since $M^{1,0}$, $M^{1,2}$, and A_H are in the image of Φ, we moreover get that Φ is surjective. This proves the first implication.

Lemma 4.38 (Lemma 7.4 in [22])**.** *Let G and H be graphs such that $G \cong_q H$. Then there exists a weak isomorphism $\Phi\colon C_q^G \to C_q^H$ between $\mathrm{Qut}(G)$ and $\mathrm{Qut}(H)$.*

A weak isomorphism conveniently abstracts the properties that we need to prove that G and H are planar isomorphic. The first observation is that weak isomorphisms mirror constructions of elements from the generators of C_q^G in C_q^H. In particular this holds for the spanning set $\{\mathbf{K}_G \mid \mathbf{K} \in \langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A}\rangle_{\circ,\otimes,*}\} \subseteq C_q^G$, as one can show by induction.

Lemma 4.39 (Lemma 7.5 in [22])**.** *Let G and H be graphs that admit a weak isomorphism $\Phi\colon C_q^G \to C_q^H$. Then for all $\mathbf{K} \in \langle \mathbf{M}^{1,0}, \mathbf{M}^{1,2}, \mathbf{A}\rangle_{\circ,\otimes,*}$ it holds that $\Phi(\mathbf{K}_G) = \mathbf{K}_H$.*

Now the elements of $C_q^G(0,0)$ are just complex multiples of $I^{\otimes 0} \in \mathbb{C}^{1\times 1}$ and one can show using a similar inductive approach as before that $\Phi(I^{\otimes 0}) = I^{\otimes 0}$. The linearity of weak isomorphisms then implies that Φ is the identity on $C_q^G(0,0)$.

Lemma 4.40. *Let G and H be graphs that admit a weak isomorphism $\Phi\colon C_q^G \to C_q^H$. Then $\Phi(I^{\otimes 0}) = I^{\otimes 0}$. In particular, the restriction of Φ to C_q^G is the identity map.*

Together, the two lemmas allow us to prove the second implication.

Lemma 4.41. *Let G and H be graphs that admit a weak isomorphism $\Phi\colon C_q^G \to C_q^H$ between $\mathrm{Qut}(G)$ and $\mathrm{Qut}(H)$. Then G and H are planar isomorphic.*

Proof. Let F be a planar graph. Then $F = \mathbf{F} \in \boldsymbol{\mathcal{P}}(0,0)$ and thus by Theorem 4.33, $\mathbf{F}_G = (\hom(F,G)) \in C_q^G(0,0)$. We thus get

$$\hom(F,G) = \mathrm{soe}(\mathbf{F}_G) = \mathrm{soe}(\Phi(\mathbf{F}_G)) = \mathrm{soe}(\mathbf{F}_H) = \hom(F,H),$$

where the second equality follows from Lemma 4.40 and the third follows from Lemma 4.39. Consequently, G and H are planar isomorphic. □

4.4.2 Planar Isomorphism implies Quantum Isomorphism

The final implication requires the most work. The general approach is to prove that if G and H are planar isomorphic, then there exists an orbit of the quantum automorphism group of $G \cup H$ that intersects both G and H non-trivially. If G and H are connected, this is equivalent by Lemma 4.5 to G and H being quantum isomorphic. We first establish some technical results that allow us to assume without loss of generality that G and H are connected.

The first lemma shows that both planar isomorphism and quantum isomorphism capture the number of connected components. In particular, both relations capture connectedness.

Lemma 4.42 (Lemma 7.11 in [22])**.** *Let G and H be quantum isomorphic graphs. Then G and H have the same number of connected components. The same holds if G and H are planar isomorphic.*

Proof. It was shown in [19] that quantum isomorphic graphs are not distinguished by 2-WL, the 2-dimensional Weisfeiler-Leman algorithm. Similarly, it is known by [8] that two graphs are not distinguished by 2-WL if $\hom(F, G) = \hom(F, H)$ for all graphs F of treewidth at most 2, and one can show that these graphs are all planar. Since 2-WL distinguishes graphs with different numbers of connected components, the lemma follows. □

Now it is a basic result that at least one of a graph or its complement is connected, and the following two lemmas allow us to simply reason about the complements of G and H if necessary.

Lemma 4.43. *Let G and H be graphs. Then G and H are quantum isomorphic if and only if their complements $\overline{G}$ and $\overline{H}$ are quantum isomorphic.*

Proof. For any graph X, we have $A_{\overline{X}} = J - I - A_X$, where J is the all-ones matrix. Per definition, G and H are quantum isomorphic if and only if there exists a quantum permutation matrix $\mathcal{U}$ with $\mathcal{U}A_G = A_H\mathcal{U}$. Since $\mathcal{U}$ commutes with J and I, this is equivalent to $\mathcal{U}A_{\overline{G}} = \mathcal{U}(J - I - A_G) = (J - I - A_H)\mathcal{U} = A_{\overline{H}}\mathcal{U}$ and hence to $\overline{G}$ and $\overline{H}$ being quantum isomorphic. □

Lemma 4.44 (Lemma 7.12 in [22])**.** *Let G and H be graphs. Then G and H are planar isomorphic if and only if their complements $\overline{G}$ and $\overline{H}$ are planar isomorphic.*

The second lemma is a direct consequence of a theorem by Seppelt [29, Theorem 8]. Given these three lemmas, we may assume without loss of generality that G and H are connected, planar isomorphic graphs. We now want to show that there exist $v \in V(G)$ and $w \in V(H)$ that are in the same orbit of $\mathrm{Qut}(G \cup H)$. We can phrase this condition in terms of homomorphism counts.

Lemma 4.45. *Let X be a graph and $u, v \in V(X)$. Then u, v are in the same orbit of $\mathrm{Qut}(X)$ if and only if*

$$(\mathbf{K}_X)_u = (\mathbf{K}_X)_v$$

for all $\mathbf{K} \in \mathcal{P}(1, 0)$.

Proof. By Lemma 4.6, $T \in C_q^X(1, 0)$ if and only if T is constant on the orbits of $\mathrm{Qut}(X)$. Therefore the forward direction is immediate. On the other hand if u, v are in distinct orbits, the characteristic vector R of the orbit containing u satisfies $R_u \neq R_v$. Since R is constant on the orbits of $\mathrm{Qut}(X)$, we have $R \in C_q^X(1, 0)$. But $R \in \mathrm{span}\{\mathbf{K}_X \mid \mathbf{K} \in \mathcal{P}(1, 0)\}$, so there must exist a $\mathbf{K} \in \mathcal{P}(1, 0)$ with $(\mathbf{K}_X)_u \neq (\mathbf{K}_X)_v$. □

The next observation shows that it really suffices to consider the connected graphs in $\mathcal{P}(1, 0)$.

Lemma 4.46 (Lemma 6.11 in [22])**.** *For any graph X,*

$$C_q^X(1, 0) = \mathrm{span}\{\mathbf{K}_X \mid \mathbf{K} \in \mathcal{P}(1, 0) \text{ connected}\}.$$

Proof. Suppose $\mathbf{K} = (K, (a), ()) \in \mathcal{P}(1, 0)$ is not connected. Let $\mathbf{K}^a = (K^a, (a), ())$ denote the connected component containing a and let K' be the graph obtained from K by removing K^a. Then $\mathbf{K}_X = \hom(K', X) \cdot \mathbf{K}_X^a$. □

Now consider some vector $T \in \mathbb{C}^{V(G)}$ and let $R \in \mathbb{C}^{V(G)}$ be the characteristic vector of an orbit of $\mathrm{Qut}(G)$. Suppose we want to extract the entries of v corresponding to the vertices belonging to this orbit. We can achieve this by computing the *Schur product* $R \bullet T$, which is defined as entrywise multiplication, $(R \bullet T)_v = R_v \cdot T_v$ for all $v \in V(G)$. In our case these vectors are given as homomorphism tensors, so we would like to define an analogous operation on bilabelled graphs.

Definition 4.47. Let $\mathbf{K} = (K, \mathbf{a}, \mathbf{b})$ and $\mathbf{K}' = (K', \mathbf{a}', \mathbf{b}')$ be two (k, ℓ)-bilabelled graphs. The *Schur product* $\mathbf{K} \bullet \mathbf{K}'$ is defined as $(K'', (a_1 a_1', \ldots, a_k a_k'), (b_1 b_1', \ldots, b_\ell b_\ell'))$, where K'' is obtained from $K \cup K'$ by identifying a_i, a_i' and b_j, b_j'—yielding the vertices $a_i a_i'$ and $b_i b_i'$—for all $i \in [k], j \in [\ell]$ and removing any multiple edges but keeping the loops.

Now one can show analogously to Lemma 4.15 that the Schur product of bilabelled graphs corresponds to that of their homomorphism tensors.

Lemma 4.48. *Let $\mathbf{K}, \mathbf{K}' \in \mathcal{G}(k, \ell)$. Then for all graphs G, we have $(\mathbf{K} \bullet \mathbf{K}')_G = \mathbf{K}_G \bullet \mathbf{K}'_G$.*

One can also show that for small values of k, ℓ the Schur product of graphs can be simulated with the help of certain planar gadget graphs using tensor products, composition, and adjoints. This implies the following lemma.

Lemma 4.49 (Lemma 6.10 in [22])**.** *For $k + \ell \leq 2$, we have that $\mathcal{P}(k, \ell)$ is closed under Schur product.*

Using these tools, Mančinska and Roberson prove the following technical lemma. The proof does not provide much topical insight, so we omit it here.

Lemma 4.50 (Lemma 7.14 in [22])**.** *Let G and H be planar isomorphic graphs and let $\mathbf{K}^{(i)} \in \mathcal{P}(1, 0)$. for $i \in [r]$. If $T = \sum_i c_i \mathbf{K}_G^{(i)}$ and $T' = \sum_i c_i \mathbf{K}_H^{(i)}$, then T and T' have the same multiset of entries.*

The lemma allows us to prove the much more interesting result that if G and H are planar isomorphic then the orbits of $\mathrm{Qut}(G)$ and $\mathrm{Qut}(H)$ look the same.

Lemma 4.51 (Lemma 7.15 in [22])**.** *Let G and H be planar isomorphic graphs. Suppose that $T_i = \sum_j c_{i,j} \mathbf{K}_G^{(i,j)}$ are the characteristic vectors of the orbits of $\mathrm{Qut}(G)$, where $\mathbf{K}^{(i,j)} \in \mathcal{P}(1, 0)$. Then $T_i' = \sum_j c_{i,j} \mathbf{K}_H^{(i,j)}$ are the characteristic vectors of the orbits of $\mathrm{Qut}(H)$.*

Proof. We first show that $T_i' \bullet T_j' = \delta_{ij} T_i'$. By Lemma 4.50, T_i and T_i' have the same multiset of entries. Since the T_i are 01-vectors, so are the T_i'. This implies that $T_i' \bullet T_i' = T_i'$. For the case $i \neq j$, note that

$$T_i \bullet T_j = \sum_{\ell k} c_{i,\ell} c_{j,k} (\mathbf{K}^{(i,\ell)} \bullet \mathbf{K}^{(j,k)})_G$$
$$T_i' \bullet T_j' = \sum_{\ell k} c_{i,\ell} c_{j,k} (\mathbf{K}^{(i,\ell)} \bullet \mathbf{K}^{(j,k)})_H.$$

Since $\mathbf{K}^{(r,s)} \in \mathcal{P}(1, 0)$, we have by Lemma 4.49 that $\mathbf{K}^{(i,\ell)} \bullet \mathbf{K}^{(j,k)} \in \mathcal{P}(1, 0)$. This in turn implies that $T_i \bullet T_j$ and $T_i' \bullet T_j'$ have the same multiset of entries. Since $T_i \bullet T_j$ is the zero vector, we get that $T_i' \bullet T_j' = \delta_{ij} T_i'$ as desired.

Now note that by Lemma 4.6, the T_i' must be constant on the orbits of $\mathrm{Qut}(H)$. This means that the T_i' are sums of one or more characteristic vectors of orbits of $\mathrm{Qut}(H)$. Since $T_i' \bullet T_j' = \delta_{ij} T_i'$, each characteristic vector occurs in only one of these sums, hence we conclude that $\mathrm{Qut}(H)$ has at least as many orbits as $\mathrm{Qut}(G)$. By symmetry, we get that $\mathrm{Qut}(G)$ and $\mathrm{Qut}(H)$ in fact have the same number of orbits. It follows that the T_i' themselves are the characteristic vectors of orbits of $\mathrm{Qut}(H)$. □

This allows us to show the final implication needed to prove Theorem 4.37.

Lemma 4.52. *Let G and H be planar isomorphic graphs. Then G and H are quantum isomorphic.*

Proof. By Lemmas 4.42 to 4.44 and the fact that either a graph or its complement is connected, we may assume that G and H are connected. Let

$$R = \sum_i c_i \mathbf{K}_G^{(i)} \in C_q^G(1, 0)$$

be the characteristic vector of an orbit of $\mathrm{Qut}(G)$. By Lemma 4.46, we may assume that the $\mathbf{K}^{(i)}$ are connected. By Lemma 4.51, we get that

$$R' = \sum_i c_i \mathbf{K}_H^{(i)} \in C_q^H(1,0)$$

is the characteristic vector of an orbit of $\mathrm{Qut}(H)$. Now let $u \in V(G)$ and $v \in V(H)$ such that $R_u = R'_v = 1$. Let $\mathbf{K} = (K, (a), ())$ be an arbitrary connected graph in $\boldsymbol{\mathcal{P}}(1,0)$ and let $X = G \cup H$. We prove that $(\mathbf{K}_X)_u = (\mathbf{K}_X)_v$.

First note that since $\mathbf{K}$ is connected, we have that for any homomorphism $h\colon \mathbf{K} \to (X, (u), ())$ the image of $V(\mathbf{K})$ is entirely contained in $V(G)$. Similarly, if h is a homomorphism $\mathbf{K} \to (X, (v), ())$, then the image of $V(\mathbf{K})$ is entirely contained in $V(H)$. We thus have

$$(\mathbf{K}_X)_u = (\mathbf{K}_G)_u \qquad \text{and} \qquad (\mathbf{K}_X)_v = (\mathbf{K}_H)_v.$$

Now consider

$$R \bullet \mathbf{K}_G = \sum_i c_i \big(\mathbf{K}^{(i)} \bullet \mathbf{K}\big)_G$$
$$R' \bullet \mathbf{K}_H = \sum_i c_i \big(\mathbf{K}^{(i)} \bullet \mathbf{K}\big)_H.$$

since $\mathbf{K}_G$ is constant on the orbits of $\mathrm{Qut}(G)$ and similarly for $\mathbf{K}_H$, we have that $R \bullet \mathbf{K}_G = \alpha R$ and $R' \bullet \mathbf{K}_H = \alpha' R'$, where $\alpha = (\mathbf{K}_G)_u$ and $\alpha' = (\mathbf{K}_H)_v$. Moreover, we can conclude from Lemma 4.50 that $\alpha \operatorname{soe}(R) = \operatorname{soe}(R \bullet \mathbf{K}_G) = \operatorname{soe}(R' \bullet \mathbf{K}_H) = \alpha' \operatorname{soe}(R')$, as well as $\operatorname{soe}(R) = \operatorname{soe}(R')$. Since $\operatorname{soe}(R) \neq 0$, it follows that $\alpha = \alpha'$ and hence

$$(\mathbf{K}_X)_u = (\mathbf{K}_G)_u = \alpha = \alpha' = (\mathbf{K}_H)_v = (\mathbf{K}_X)_v$$

as desired. Finally, Lemma 4.46 implies that this even holds for non-connected graphs $\mathbf{K}$, so by Lemma 4.45, u and v are in the same orbit of $\mathrm{Qut}(G \cup H)$. By Lemma 4.5, it follows that G and H are quantum isomorphic. □

Together, Lemmas Lemmas 4.38, 4.41 and 4.52 prove Theorem 4.37. We have thus shown that quantum isomorphism is equivalent to homomorphism indistinguishability over the class of all planar graphs.

5 Beyond the Quantum Symmetric Group

> Now is no time to think of what you do not have. Think of what you can do with that there is.
>
> *Ernest Hemingway, The Old Man and the Sea*

The core of Mančinska and Roberson's proof is to reason about the quantum automorphism group of a graph G, which is obtained by taking the intertwiners defining the quantum symmetric group S_n^+ and additionally requiring that A_G be an intertwiner. The connection to homomorphism counts and planar graphs stems from the fact that the intertwiners of a $\mathrm{Qut}(G)$ can be characterised in terms of homomorphism tensors of certain graphs associated to the quantum group. Specifically, these graphs are generated by the edgeless bilabelled graphs corresponding to the generators of the quantum symmetric group, together with the bilabelled edge, whose homomorphism tensors are adjacency matrices. The resulting graph class then turns out to be precisely the class of planar bilabelled graphs.

The central results and techniques involved are by no means unique to the quantum symmetric group. In fact, the intertwiners of any easy quantum group Q can be characterised in terms of homomorphism tensors of edgeless bilabelled graphs. Adding the bilabelled edge to these intertwiner graphs then induces a graph category, which in turn gives rise to a class of "graph-specific" compact matrix quantum subgroups of Q, analogous to what the quantum automorphism groups are to the quantum symmetric group.

In the light of the complete classification of easy quantum groups, this raises the question to what extent Mančinska and Roberson's argument can be generalised to the other easy quantum groups to obtain new homomorphism indistinguishability characterisations. The most obvious hurdle to this is the proof of the backward direction, which uses very specific properties of quantum isomorphism and planar graphs. This poses the question whether such an argument can be replaced by a more general approach on the level of intertwiners, or if the backward direction has to be argued individually for each quantum group.

In this chapter, we answer these questions. We prove a general theorem relating quantum groups to homomorphism indistinguishability relations over their intertwiner graphs. In particular, we succeed in generalising the backward direction of Mančinska and Roberson's proof, avoiding the need to reason about individual homomorphism indistinguishability relations, albeit at the cost of substantial category-theoretic machinery. Along the way, we determine the graph categories of the intertwiner graphs of easy quantum groups, resulting in a full classification of their $(0,0)$-fragments. We conclude by stating some important corollaries in a simplified form.

5.1 The Graph Instantiation

We begin by generalising the definitions inherent to Mančinska and Roberson's proof. The first concept we need to formalise is the idea of "adding the bilabelled edge". For brevity, we simplify some definitions. Let Q be an easy quantum group corresponding to some partition category $\langle P_1, \dots, P_k \rangle$. Then we call the linear maps $I, M^{2,0}, M_1, \dots, M_k$ corresponding to $\mid, \sqcap, P_1, \dots, P_k$ the *generators* of Q. Moreover, we denote the corresponding edgeless bilabelled graphs by bold letters, in this case $\mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}_1, \dots, \mathbf{M}_k$.

Definition 5.1. Let Q be an easy quantum group with generators $M_1, \dots, M_k$ and let G be a graph. Then the *G-instantiation* $Q(G)$ of Q is the unique compact matrix quantum group with intertwiners $C_G^Q(k,\ell) := \operatorname{span}\{\mathbf{K}_G \mid \mathbf{K} \in \langle \mathbf{M}_1, \dots, \mathbf{M}_k, \mathbf{A}\rangle_{\circ,\otimes,*} \cap \mathcal{G}(k,\ell)\}$.

In particular, the G-instantiation of S_n^+ is simply the quantum automorphism group $\operatorname{Qut}(G)$ and we have $C_G^{S_n^+} = C_q^G = C^{\operatorname{Qut}(G)}$. Central to the proof is to determine which graphs are generated by $\mathbf{M}_1, \dots, \mathbf{M}_k, \mathbf{A}$. Since these graphs only depend on the easy quantum group Q, we also call them the *intertwiner graphs* of Q.

Definition 5.2. Let Q be an easy quantum group with generators $M_1, \dots, M_k$. We call the set of graphs $\mathcal{F} = \langle \mathbf{M}_1, \dots, \mathbf{M}_k, \mathbf{A}\rangle_{\circ,\otimes,*}$ the *intertwiner graphs* of Q. Moreover, for all $k, \ell \in \mathbb{N}$ we denote by $\mathcal{F}(\ell,k) \subseteq \mathcal{F}$ the set of (ℓ,k)-bilabelled graphs in $\mathcal{F}$.

The intertwiner graphs of the quantum symmetric group, for instance, are exactly the planar bilabelled graphs. Next, we define a shorthand notation for homomorphism indistinguishability over the unlabelled graphs in a graph category.

Definition 5.3. Let $\mathcal{F}$ be a collection of bilabelled graphs. We say that two graphs G, H are *$\mathcal{F}$-isomorphic*, in symbols $G \cong_{\mathcal{F}} H$, if G and H are homomorphism indistinguishable over $\mathcal{F}(0,0)$, that is,

$$\hom(F,G) = \hom(F,H)$$

for all $F \in \mathcal{F}(0,0)$.

Under these definitions, Mančinska and Roberson's theorem states that two graphs G and H are quantum isomorphic if and only if they are $\mathcal{P}$-isomorphic. A central definition of its proof are weak isomorphisms between $\operatorname{Qut}(G)$ and $\operatorname{Qut}(H)$. These functions essentially capture that the intertwiner spaces of the two graph instantiations have the same structure. In particular, every construction of an intertwiner from the generators of $\operatorname{Qut}(G)$ can be mirrored in $\operatorname{Qut}(H)$. It is fairly straightforward to generalise this to graph instantiations of arbitrary easy quantum groups.

Definition 5.4. Let $Q = (C(Q), \mathcal{U}, n)$ be an easy quantum group with generators $M_1, \dots, M_k$, and let G, H be two graphs of order n. A *weak isomorphism* between $Q(G)$ and $Q(H)$ is a bijection $\Phi\colon C_G^Q \to C_H^Q$ such that for all $T, T' \in C_G^Q$,

(i) $\Phi(A_G) = A_H$ and $\Phi(M_i) = M_i$ for all $i \in [k]$,

(ii) $\Phi(TT') = \Phi(T)\Phi(T')$ if T, T' are compatible,

(iii) $\Phi(T \otimes T') = \Phi(T) \otimes \Phi(T')$,

(iv) $\Phi(T^*) = \Phi(T)^*$,

(v) For all $\ell, k \in \mathbb{N}$, the restriction of Φ to $C_G^Q(\ell,k)$ is a bijective linear map to $C_H^Q(\ell,k)$.

Remark 5.5. Technically, we should write $\Phi_{M_1,\dots,M_k}$ since the definition depends on the choice of generators. However, it will later become clear that for any two choices of generators $M_1, \dots, M_k$ and $M_1', \dots, M_\ell'$ we have that $\Phi(M_i) = M_i$ for all $i \in [k]$ implies $\Phi(M_j') = M_j'$ for all $j \in [\ell]$.

For easy quantum groups, we can always assume that $M^{2,0}$ is among the generators. This allows us to prove the following analogue of Lemma 4.40, which fixes the $(0,0)$-slice of any weak isomorphism.

Lemma 5.6. *Let $(C(Q), \mathcal{U}, n)$ be an easy quantum group, and G, H be graphs that admit a weak isomorphism $\Phi\colon C_G^Q \to C_H^Q$. Then $\Phi(I^{\otimes 0}) = I^{\otimes 0}$. In particular, the restriction of Φ to $C_G^Q(0,0)$ is the identity map.*

Proof. Per definition of weak isomorphisms, we have $|G| = |H| =: n$. We use that the map $M := M^{2,0} \in C_G^Q$ satifies $M^*M = nI^{\otimes 0}$. Then we get

$$n\Phi(I^{\otimes 0}) = \Phi(nI^{\otimes 0}) = \Phi(M^*M) = \Phi(M^*)\Phi(M) = M^*M = nI^{\otimes 0}.$$

Dividing by n on both sides gives us the desired result. The linearity of Φ then further implies that its restriction to $C_G^Q(0,0)$ must be the identity map. □

5.2 Cycles and More: Constructing Intertwiner Graphs

We have seen that the fundamental concepts of Mančinska and Roberson's proof are readily generalised to arbitrary easy quantum groups. In line with their proof, we continue by investigating the intertwiner graphs $\mathcal{F}$ of the different easy quantum groups. While in Chapter 4 we identified the entire class of intertwiner graphs, we will here only focus on the subclass $\mathcal{F}(0,0)$ of unlabelled graphs in $\mathcal{F}$. This simplifies our presentation somewhat, and is sufficient to state the resulting homomorphism indistinguishability characterisations. To structure this section, we follow the classification of Theorem 3.65.

In the following we understand all mentioned classes to be closed under disjoint unions. This is because for any two constructed graphs $\mathbf{K}, \mathbf{K}'$, their disjoint union is $\mathbf{K} \otimes \mathbf{K}'$, and thus always constructible. In the context of homomorphism indistinguishability this makes no difference, since $\hom(F_1 \cup F_2, G) = \hom(F_1, G) \cdot \hom(F_2, G)$ for all graphs F_1, F_2, G. For brevity, we refrain from drawing all the involved intermediate and gadget graphs, and only highlight a few central ones in Figures 5.1 and 5.4. However, most of the constructions should be clear from a quick sketch on a piece of paper.

5.2.1 Classical, Liberated, and Half-Liberated Quantum Groups

Let us begin by investigating the classical, liberated, and the half-liberated quantum groups—the latter being the easy quantum groups $G_n(\mathcal{C})$ with $\diagup\!\!\diagdown \in \mathcal{C}$ and $\diagup\!\diagup \notin \mathcal{C}$.

The first observation is that the bilabelled graphs $\mathbf{I}$ and $\mathbf{M}^{2,0}$ already allow us to construct arbitrary cycles. Since every easy quantum group has I and $M^{2,0}$ as intertwiners, their corresponding intertwiner graphs always include all cycles. In particular, the class of unlabelled intertwiner graphs of the quantum orthogonal group—which is generated precisely by I and $M^{2,0}$—is exactly the class of all cycles. Moreover, the swap graph $\mathbf{S}$ does not allow us to construct any additional graphs, so the same holds for the classical orthogonal group, generated by $I, M^{2,0}, S$.

Lemma 5.7. *Let $\mathcal{F}$ be the class of intertwiner graphs of the orthogonal group and $\mathcal{F}^+$ be the class of intertwiner graphs of the quantum orthogonal group. Then $\mathcal{F}(0,0) = \mathcal{F}^+(0,0) = \mathcal{C}$, the class of all cycles.*

Proof. We have $\mathcal{F} = \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{S}, \mathbf{A} \rangle_{\circ,\otimes,*}$ and $\mathcal{F}^+ = \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{A} \rangle_{\circ,\otimes,*}$. We first show that we can construct arbitrary cycles. We have $C_1 = \mathbf{M}^{0,2}\mathbf{M}^{2,0}$ and for $n \geq 2$,

$$C_n = \mathbf{M}^{0,2}(\mathbf{A} \otimes \mathbf{A}^{n-1})\mathbf{M}^{2,0}.$$

To see that $\mathcal{F}(0,0) \supseteq \mathcal{F}^+(0,0)$ only contains cycles, we show that every vertex has degree 2, or is part of either C_1 or C_2. For finite graphs, this is equivalent to being a disjoint union of cycles. Let $\mathbf{K} \in \mathcal{F}$ and denote for $\mathbf{K}^v$ the connected component containing $v \in V(\mathbf{K})$. By induction, we show the slightly stronger statement that all vertices $v \in V(\mathbf{K})$ satisfy either $\deg(v) + |l(v)| = 2$, $\mathbf{K}^v = C_1$, or $\mathbf{K}^v = C_2$.

Obviously this holds for the generators $\mathbf{I}, \mathbf{M}^{2,0}, \mathbf{S}, \mathbf{A}$, so assume it holds for $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{F}$. The inductive step is trivial for $\mathbf{K} = \mathbf{K}_1 \otimes \mathbf{K}_2$ and $\mathbf{K} = \mathbf{K}_1^*$, so assume $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$. The composition

leaves unlabelled vertices unchanged, and by assumption identifies at most 3 vertices at a time. Let $v, v_1, v_2 \in \mathbf{K}_1$ and $w \in \mathbf{K}_2$. We make the following case distinction.

$|l_{out}(v)| = |l_{in}(w)| = 2$: In this case v and w must be isolated vertices, and thus their identification vw is an isolated vertex without labels. Consequently, $\mathbf{K}^{vw} = C_1$.

$|l_{out}(v)| = |l_{in}(w)| = 1$: In this case v and w have exactly one adjacent vertex in $\mathbf{K}_1$ and $\mathbf{K}_2$ respectively, so vw is a labelless vertex of degree 2.

$|l_{out}(v_1)| = |l_{out}(v_2)| = 1$, $|l_{in}(w)| = 2$: In this case w is an isolated vertex, and v_1, v_2 both have exactly one adjacent vertex in $\mathbf{K}_1$. Denote the adjacent vertices by v_1' and v_2' respectively. If $v_1' \neq v_2'$, then $v_1 v_2 w$ is a labelless vertex of degree 2 as desired. If $v_1' = v_2' := v'$, then $\deg(v') = 2$. By assumption this implies that v' is labelless and has no other neighbours. Consequently, $\mathbf{K}^{v'} = \mathbf{K}^{v_1 v_2 w} = C_2$.

The remaining case is symmetric. □

Obviously, this also holds for quantum groups Q with $O_n \subseteq Q \subseteq O_n^+$.

Corollary 5.8. *Let $\mathcal{F}$ be the class of intertwiner graphs of $G_n(E_o)$. Then $\mathcal{F}(0,0) = \mathcal{C}$, the class of all cycles.*

Proof. We have $E_o = \langle \times\!\!\times \rangle$. Let $\bar{\mathbf{S}}$ denote the bilabelled graph corresponding to $\times\!\!\times$. Then $\mathcal{F} = \langle \mathbf{I}, \mathbf{M}^{2,0}, \bar{\mathbf{S}} \rangle_{\circ,\otimes,*}$, and we have

$$\bar{\mathbf{S}} = (\mathbf{S} \otimes \mathbf{I}) \circ (\mathbf{I} \otimes \mathbf{S}) \circ (\mathbf{S} \otimes \mathbf{I}).$$

In particular, we get $\mathcal{C} = \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{A} \rangle_{\circ,\otimes,*} \subseteq \langle \mathbf{I}, \mathbf{M}^{2,0}, \bar{\mathbf{S}}, \mathbf{A} \rangle_{\circ,\otimes,*} \subseteq \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{S}, \mathbf{A} \rangle_{\circ,\otimes,*} = \mathcal{C}$. □

Along the same lines as Lemma 5.7, we can prove the following result for the bistochastic and quantum bistochastic groups.

Lemma 5.9. *Let $\mathcal{F}$ be the class of intertwiner graphs of the bistochastic group and $\mathcal{F}^+$ be the class of intertwiner graphs of the quantum bistochastic group. Then $\mathcal{F}(0,0) = \mathcal{F}^+(0,0)$ is the class of all paths and cycles.*

Proof. We have $\mathcal{F} = \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{1,0}, \mathbf{S}, \mathbf{A} \rangle_{\circ,\otimes,*}$ and $\mathcal{F}^+ = \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{1,0}, \mathbf{A} \rangle_{\circ,\otimes,*}$. We proceed similarly to the previous lemma. We already know how to construct arbitary cycles. P_1 is again equal to $\mathbf{M}^{0,2}\mathbf{M}^{2,0}$, and for $n \geq 2$, we have

$$P_n = \mathbf{M}^{0,1} \circ \mathbf{A}^{n-1} \circ \mathbf{M}^{1,0}.$$

To see that $\mathcal{F}(0,0) \supseteq \mathcal{F}^+(0,0)$ only contains paths and cycles, we show that every vertex has degree at most 2. For finite graphs, this is equivalent to being a disjoint union of paths and cycles. We show the slightly stronger statement that for all $\mathbf{K} \in \mathcal{F}$ all vertices $v \in V(K)$ satisfy $\deg(v) + |l(v)| \leq 2$.

Obviously, this holds for $\mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{1,0}, \mathbf{S}$, so assume it holds for $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{F}$. The inductive step is again trivial for $\mathbf{K} = \mathbf{K}_1 \otimes \mathbf{K}_2$ and $\mathbf{K} = \mathbf{K}_1^*$, so assume $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$. The composition leaves unlabelled vertices unchanged and by assumption identifies at most 3 vertices at a time. Let $v, v_1, v_2 \in \mathbf{K}_1$ and $w \in \mathbf{K}_2$. We make the following case distinction.

$|l_{out}(v)| = |l_{in}(w)| = 2$: In this case v and w must be isolated vertices, and thus their identification vw is an isolated vertex without labels. In particular, $\deg(vw) + |l(vw)| = 0$.

$|l_{out}(v)| = |l_{in}(w)| = 1$: In this case v and w have at most one adjacent vertex in $V(K_1)$ and $V(K_2)$ respectively. It follows that vw is an unlabelled vertex with degree $\deg(v) + \deg(w) \leq 2$.

$|l_{out}(v_1)| = |l_{out}(v_2)| = 1$, $|l_{in}(w)| = 2$: In this case w is an isolated vertex, and v_1, v_2 both have at most one adjacent vertex in $\mathbf{K}_1$. Thus $v_1 v_2 w$ is an unlabelled vertex with $\deg(v_1 v_2 w) \leq \deg(v_1) + \deg(v_2) \leq 2$.

The remaining case is symmetric. □

Since paths are fairly easy to construct, we obtain the following result for supergroups of the (quantum) bistochastic group.

Lemma 5.10. *Let Q be an easy quantum group and $\mathcal{F}$ the class of its intertwiner graphs. If Q is either*

(*i*) *the bistochastic group B_n,*

(*ii*) *the quantum bistochastic group B_n^+,*

(*iii*) *the modified bistochastic group B_n',*

(*iv*) *the quantum modified bistochastic group $B_n^{\#+}$,*

(*v*) *the quantum complexly modified bistochastic group $B_n'^+$,*

(*vi*) *$G_n(E_b')$,*

then $\mathcal{F}(0,0)$ is the class of all paths and cycles.

Proof. The cases (i) and (ii) are proved in Lemma 5.9. For the remaining groups, their defininig partition categories are either subsets of those defining the bistochastic group or those defining the quantum bistochastic group (depending on whether they contain crossing partitions). This means that their intertwiner graphs must be disjoint unions of paths or cycles. Moreover, the partition categories are obviously supersets of those defining the orthogonal group, so we can construct arbitrary cycles. Consequently, it suffices to show that we can also construct arbitrary paths.

For B_n', $B_n^{\#+}$, and $G_n(E_b')$ we have $\mathbf{M}^{1,0} \otimes \mathbf{M}^{1,0} \in \mathcal{F}$. This allows us to construct P_n as

$$P_n = (\mathbf{M}^{0,1} \otimes \mathbf{M}^{0,1}) \circ (\mathbf{A}^n \otimes \mathbf{I}) \circ \mathbf{M}^{2,0}.$$

For $B_n'^+$, we instead have access to $\mathbf{M}^{1,0} \otimes \mathbf{I} \otimes \mathbf{M}^{0,1}$, which allows us to construct P_n as

$$\mathbf{M}^{0,2} \circ (\mathbf{M}^{1,0} \otimes \mathbf{I} \otimes \mathbf{M}^{0,1}) \circ (\mathbf{A}^n \otimes \mathbf{I}) \circ \mathbf{M}^{2,0}. \qquad \square$$

The remaining easy quantum groups enjoy much larger classes of intertwiner graphs. This is in part due to a general construction that allows us to compose any unlabelled graph when we have access to $\mathbf{M}^{2,2}$ and some "swapping" partition.

Lemma 5.11. *Let $\mathcal{G}$ denote the class of all (unlabelled) graphs. Then $\mathcal{G} \subseteq \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{2,2}, \mathbf{S} \rangle_{\circ,\otimes,*}$ and $\mathcal{G} \subseteq \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{2,2}, \bar{\mathbf{S}} \rangle_{\circ,\otimes,*}$.*

Proof. We first show that $\mathcal{G} \subseteq \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{2,2}, \bar{\mathbf{S}} \rangle_{\circ,\otimes,*}$. Let $G \in \mathcal{G}$ with $|G| =: n$. We construct G as follows. Fix some order $v_1, \dots, v_n$ on $V(G)$ and start with n copies of $\mathbf{M}^{2,0}$ that we identify with the vertices. We now construct two gadgets. The first, $\hat{\mathbf{A}}$, allows us to add an edge between vertices v_i and v_{i+1} for all $i \in [n-1]$. First note that $\mathbf{M}^{4,2} = (\mathbf{M}^{2,2} \otimes \mathbf{M}^{2,2}) \circ (\mathbf{I} \otimes \mathbf{M}^{2,0} \otimes \mathbf{I})$. We let

$$\hat{\mathbf{A}} = (\mathbf{I}^{\otimes 2} \otimes \mathbf{M}^{2,4}) \circ (\mathbf{I}^{\otimes 2} \otimes \mathbf{A}^{\otimes 2} \otimes \mathbf{I}^{\otimes 2}) \circ (\mathbf{M}^{4,2} \otimes \mathbf{I}^{\otimes 2}).$$

Composition with $\mathbf{I}^{\otimes 2(i-1)} \otimes \hat{\mathbf{A}} \otimes \mathbf{I}^{\otimes 2(n-i-1)}$ then adds an edge between v_i and v_{i+1} while preserving the out-labels. The second gadget allows us to swap any two vertices adjacent in the ordering. We let

$$\hat{\mathbf{S}} = (\mathbf{I} \otimes \bar{\mathbf{S}}) \circ (\bar{\mathbf{S}} \otimes \mathbf{I}).$$

Composition with $\hat{\mathbf{S}}_i \coloneqq \mathbf{I}^{\otimes 2(i-1)} \otimes \hat{\mathbf{S}} \otimes \mathbf{I}^{\otimes 2(n-i-1)}$ swaps v_i and v_{i+1}. More precisely, let $\mathbf{K} = (K, (v_1, v_1, \dots, v_n, v_n), ())$. Then it is straightforward to verify that

$$\hat{\mathbf{S}}_i \circ \mathbf{K} = (K, (v_1, v_1, \dots, v_{i+1}, v_{i+1}, v_i, v_i, \dots, v_n, v_n), ()).$$

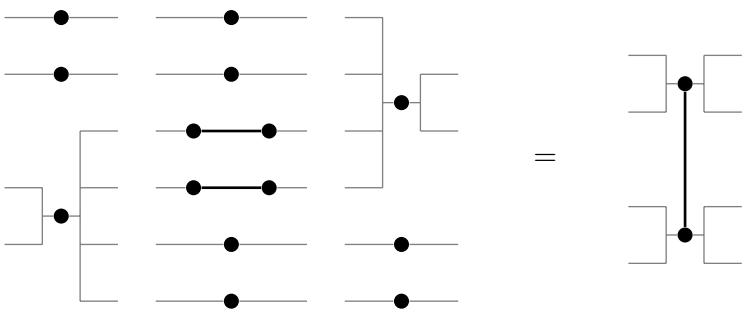


Figure 5.1: Constructing the edge gadget $\hat{\mathbf{A}}$.

Together, these two gadgets allow us to add edges between arbitrary vertices. After adding the edges, we can remove the $2n$ out-labels by composition with $(\mathbf{M}^{0,2})^{\otimes n}$. Finally, since

$$\bar{\mathbf{S}} = (\mathbf{S} \otimes \mathbf{I}) \circ (\mathbf{I} \otimes \mathbf{S}) \circ (\mathbf{S} \otimes \mathbf{I}),$$

we get that $\mathcal{G} \subseteq \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{2,2}, \bar{\mathbf{S}} \rangle_{\circ,\otimes,*} \subseteq \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{2,2}, \mathbf{S} \rangle_{\circ,\otimes,*}$. □

In the context of the classification results of Section 3.4.3, we can thus state the following corollary.

Corollary 5.12. *Let Q be an easy quantum group and $\mathcal{F}$ the class of its intertwiner graphs. If Q is either*

- *the symmetric group S_n,*
- *the hyperoctahedral group H_n,*
- *the modified symmetric group S'_n,*
- *$G_n(E_h)$,*
- *$G_n(E_h^s)$,*

then $\mathcal{F}(0,0) = \mathcal{G}$, the class of all graphs.

In the case of the quantum modified symmetric- and the quantum hyperoctahedral group the $(0,0)$-intertwiner graphs turn out to be all planar graphs. In fact, an approach analogous to Mančinska and Roberson's proof for the quantum symmetric group goes through if we replace single labels by double labels. To that end we introduce the following gadget graphs.

Definition 5.13. Let S^d be the star graph on $d+1$ vertices with vertex set $\{v, v_1, \ldots, v_d\}$ and edges $\{vv_i \mid i \in [d]\}$. For all $m, d \in \mathbb{N}$, we define ${}^2\mathbf{S}^{m,d} := (S^{2d}, (v^{2m}), (v_1, \ldots, v_{2d}))$, ${}^2\mathbf{S}_L^{m,d} := (S^{2d}, (v^{2m}), (v, v, v_1, \ldots, v_{2d}))$, and ${}^2\mathbf{S}_R^{m,d} := (S^{2d}, (v^{2m}), (v_1, \ldots, v_{2d}, v, v))$.

Recall the definition of planar bilabelled graphs through enveloping cycles. We let ${}^2\boldsymbol{\mathcal{P}}(\ell, k)$ denote the class of planar bilabelled graphs $\mathbf{K}$ of the form $(K, (a_1, a_1, \ldots, a_\ell, a_\ell), (b_1, b_1, \ldots, b_k, b_k))$. Then we can prove the following lemma similarly to Lemma 4.32.

Lemma 5.14. *Let $\mathbf{K} \in {}^2\boldsymbol{\mathcal{P}}$ with $V(\mathbf{K}) \geq 2$. Then there exists a bilabelled graph $\mathbf{K}' \in {}^2\boldsymbol{\mathcal{P}}$ with $|V(\mathbf{K}')| = |V(\mathbf{K})| - 1$ and $m, d, r, t \in \mathbb{N}$ such that one of the following holds:*

$$\mathbf{K} = (\mathbf{I}^{\otimes 2r} \otimes {}^2\mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes 2t}) \circ \mathbf{K}' \tag{5.1}$$

$$\mathbf{K} = \mathbf{K}' \circ (\mathbf{I}^{\otimes 2r} \otimes {}^2\mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes 2t})^* \tag{5.2}$$

$$\mathbf{K} = ({}^2\mathbf{S}_L^{m,d} \otimes \mathbf{I}^{\otimes 2t}) \circ (\mathbf{M}^{2,2r} \otimes \mathbf{K}') \tag{5.3}$$

$$\mathbf{K} = (\mathbf{I}^{\otimes 2t} \otimes {}^2\mathbf{S}_R^{m,d}) \circ (\mathbf{K}' \otimes \mathbf{M}^{2,2r}) \tag{5.4}$$

Proof. Suppose first that $\mathbf{K} = (K, \mathbf{a}, \mathbf{b}) \in {}^2\boldsymbol{\mathcal{P}}(\ell, k)$ with $\ell + k \geq 1$. By Corollary 4.31, there exists a $v \in V(\mathbf{K})$ that occurs consecutively in $(a_1, \dots, a_\ell, b_k, \dots, b_1)$. We will construct $\mathbf{K}'$ by removing v. We make the following case distinction.

Case 1: *v occurs only in* $\mathbf{a}$. In this case, v might occur consecutively by appearing in $a_1, \dots, a_p$ and $a_q, \dots, a_\ell$ for some $p, q \in [\ell]$. But if this is the case, we can find, by Corollary 4.31, a $v' \in V(\mathbf{K})$ that occurs consecutively and only among $a_{p+1}, \dots a_{q-1}$. We may thus assume that the occurrences of v in $\mathbf{a}$ are precisely $a_p, \dots, a_q$ for some $p, q \in [\ell]$. Now let $v_1, \dots, v_d$ be the neighbours of v, and set $\mathbf{K}' = (K', \mathbf{a}', \mathbf{b})$, where $K' = K - v$ and $\mathbf{a}' = (a_1, a_1, \dots, a_{p-1}, v_1, v_1, \dots, v_d, v_d, a_{q+1}, \dots, a_\ell)$. Then

$$\mathbf{K} = (\mathbf{I}^{\otimes 2r} \otimes {}^2\mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes 2t}) \circ \mathbf{K}',$$

where $r = p - 1$, $m = q - p + 1$, and $t = \ell - q$. The proof of the planarity of $\mathbf{K}'$ is similar to the approach in Lemma 4.32, although we need an additional step to duplicate the out-labels of $v_1, \dots, v_d$. We first proceed as in Lemma 4.32, which adds for each v_i a new adjacent vertex u_i of the enveloping cycle. We then subdivide one of the edges of the enveloping cycle incident to u_i with a new vertex u_i' and add the edge $v_i u_i'$. Lemma 4.23 then guarantees that the resulting graph is still planar. We illustrate this additional step in Figure 5.2.

Case 2: *v occurs only in* $\mathbf{b}$. In this case $\mathbf{K}^*$ falls into Case 1. Consequently, there exists a $\mathbf{K}'$ such that

$$\mathbf{K}^* = (\mathbf{I}^{\otimes 2r} \otimes {}^2\mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes 2t}) \circ \mathbf{K}'.$$

Taking the adjoint of both sides yields

$$\mathbf{K} = \mathbf{K}'^* \circ (\mathbf{I}^{\otimes 2r} \otimes {}^2\mathbf{S}^{m,d} \otimes \mathbf{I}^{\otimes 2t})^*,$$

and it is not hard to see that $\mathbf{K}'^* \in {}^2\boldsymbol{\mathcal{P}}$.

Case 3: $a_1 = b_1 = v$. In this case v occurs in both $\mathbf{a}$ and $\mathbf{b}$, so we need to do slightly more work to ensure the correct labelling. First we consider the special subcase where $a_\ell = b_k = v$, but not all entries of $\mathbf{a}$ and $\mathbf{b}$ are equal to v. In that case there exists by Corollary 4.31 a $v' \in V(\mathbf{K})$ that occurs consecutively and only among $\mathbf{a}$ or $\mathbf{b}$. Thus either Case 1 or Case 2 apply.

Otherwise, we can assume that the occurrences of v are exactly $a_1, \dots, a_m$ and $b_1, \dots, b_r$ respectively. Again let $v_1, \dots, v_d$ be the neighbours of v in $\mathbf{K}$ and set $\mathbf{K}' = (K', \mathbf{a}', \mathbf{b}')$, where $K' = K - v$, $\mathbf{a}' = (v_1, v_1, \dots, v_d, v_d, a_{m+1}, a_{m+1}, \dots, a_\ell)$, and $\mathbf{b}' = (b_{r+1}, b_{r+1}, \dots, b_k)$. Let $t = \ell - m$. Then we have

$$\mathbf{K} = ({}^2\mathbf{S}_L^{m,d} \otimes \mathbf{I}^{\otimes 2t}) \circ (\mathbf{M}^{2,2r} \otimes \mathbf{K}').$$

To see that $\mathbf{K}'$ is planar, we reduce to Case 1. Consider $\hat{\mathbf{K}} = \mathbf{K} \circ (\mathbf{M}^{2r,0} \otimes \mathbf{I}^{\otimes 2(k-r)})$. This removes the occurrences of v from $\mathbf{b}$, so it falls into Case 1. Moreover, the $\hat{\mathbf{K}}'$ constructed in Case 1 is exactly the $\mathbf{K}'$ constructed here. In particular, $\mathbf{K}' \in {}^2\boldsymbol{\mathcal{P}}$.

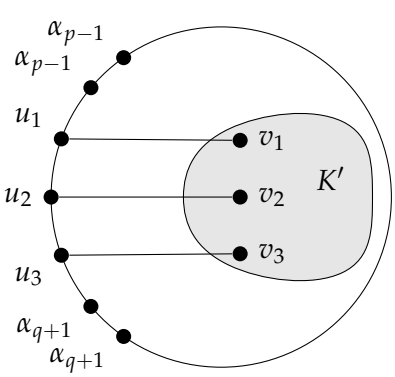
Case 4: $a_\ell = b_k = v$. This case is symmetric to Case 3.

Case 5: $k + \ell = 0$. In this case there is no enveloping cycle. We thus choose an arbitrary vertex $v \in V(\mathbf{K})$ with neighbours $v_1, \dots v_d$ and let $\mathbf{K}' = (K', (v_1, v_1, \dots, v_d, v_d), ())$ with $K' = K - v$. Then $\mathbf{K} = {}^2\mathbf{S}^{0,2d} \circ \mathbf{K}'$. To see that $\mathbf{K}'$ is planar, fix some planar embedding of K where $v_1, \dots, v_d$ appear in cyclic order. Subdivide the edges vv_i with vertices α_i and add edges to create a cycle $C = \alpha_1, \dots, \alpha_d$. By Lemma 4.24, the resulting graph is planar and is in fact isomorphic to $K'^{\odot}$. It follows that $\mathbf{K}' \in {}^2\boldsymbol{\mathcal{P}}$. □

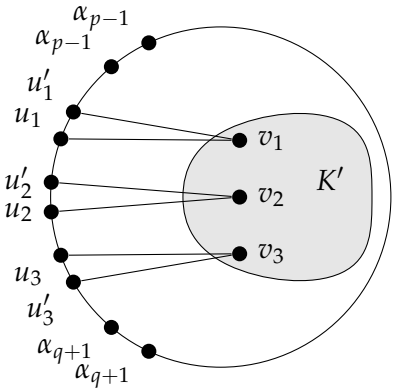
The lemma now allows us to inductively construct our desired planar graphs.

Lemma 5.15. *Let $\mathcal{F}$ be the class of intertwiner graphs of the quantum hyperoctahedral group, and $\mathcal{F}'$ be the class of intertwiner graphs of the quantum modified symmetric group. Then $\mathcal{F}(0,0) = \mathcal{F}'(0,0) = \mathcal{P}$, the class of all planar graphs.*

Proof. We show the stronger result that ${}^2\boldsymbol{\mathcal{P}} \subseteq \mathcal{F} \cap \mathcal{F}'$ by induction. The single vertex graphs in ${}^2\boldsymbol{\mathcal{P}}$ are precisely the $\mathbf{M}^{2r,2s}$ for $r, s \in \mathbb{N}$. The induction base thus follows from the next claim.



K° after the construction steps outlined in Figure 4.5



Subdivide an edge incident to u_i with vertex u_i' and adding the edge $v_i u_i'$.

Figure 5.2: Duplicating vertices on the enveloping cycle.

Claim 1. *Let $r, s \in \mathbb{N}$. Then $\mathbf{M}^{2r,2s} \in \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{2,2} \rangle_{\circ,\otimes,*}$.*

Proof. Suppose we already have $\mathbf{M}^{2r,2s}$ for some $r, s \geq 1$. Then we can construct $\mathbf{M}^{2r,2s+2}$ and $\mathbf{M}^{2r+2,2s}$. Indeed, we have

$$\mathbf{M}^{2r,2s+2} = (\mathbf{I}^{\otimes(2r-1)} \otimes \mathbf{M}^{0,2} \otimes \mathbf{I}) \circ (\mathbf{M}^{2r,2s} \otimes \mathbf{M}^{2,2}),$$
$$\mathbf{M}^{2r+2,2s} = (\mathbf{M}^{2r,2s} \otimes \mathbf{M}^{2,2}) \circ (\mathbf{I}^{\otimes(2s-1)} \otimes \mathbf{M}^{2,0} \otimes \mathbf{I}).$$

It thus follows by induction that we can construct $\mathbf{M}^{2r,2s}$ for all $r, s \geq 1$. Finally, we have $\mathbf{M}^{2r,0} = \mathbf{M}^{2r,2} \circ \mathbf{M}^{2,0}$ and $\mathbf{M}^{0,2s} = (\mathbf{M}^{2s,0})^*$, which completes the proof. ■

Now consider some $\mathbf{K} \in {}^2\boldsymbol{\mathcal{P}}$ with $V(\mathbf{K}) \geq 2$. By Lemma 5.14, there exists a bilabelled graph $\mathbf{K}' \in {}^2\boldsymbol{\mathcal{P}}$ with $|V(\mathbf{K}')| = |V(\mathbf{K})| - 1$ such that one of Equations (5.1) to (5.4) holds. By induction hypothesis we can construct $\mathbf{K}'$, so it suffices to show that we can construct ${}^2\mathbf{S}^{m,d}$, ${}^2\mathbf{S}_L^{m,d}$, and ${}^2\mathbf{S}_R^{m,d}$.

Claim 2. *Let $m, d \in \mathbb{N}$. Then ${}^2\mathbf{S}^{m,d}, {}^2\mathbf{S}_L^{m,d}, {}^2\mathbf{S}_R^{m,d} \in \langle \mathbf{I}, \mathbf{M}^{2,0}, \mathbf{M}^{2,2}, \mathbf{A} \rangle_{\circ,\otimes,*}$.*

Proof. We have seen that we can construct $\mathbf{M}^{2m,2d}$, which allows us to construct ${}^2\mathbf{S}^{m,d}$ as

$${}^2\mathbf{S}^{m,d} = \mathbf{M}^{2m,2d} \circ \mathbf{A}^{\otimes 2d}.$$

Similarly, we can construct ${}^2\mathbf{S}_L^{m,d}$ and ${}^2\mathbf{S}_R^{m,d}$ as

$${}^2\mathbf{S}_L^{m,d} = \mathbf{M}^{2m,2d+2} \circ (\mathbf{I}^{\otimes 2} \otimes \mathbf{A}^{\otimes 2d}),$$
$${}^2\mathbf{S}_R^{m,d} = \mathbf{M}^{2m,2d+2} \circ (\mathbf{A}^{\otimes 2d} \otimes \mathbf{I}^{\otimes 2}).$$ ■

We have thus shown that ${}^2\boldsymbol{\mathcal{P}} \subseteq \mathcal{F} \cap \mathcal{F}'$. In particular, this means that $\mathcal{P} = {}^2\boldsymbol{\mathcal{P}}(0,0) \subseteq \mathcal{F}(0,0) \cap \mathcal{F}'(0,0) \subseteq \mathcal{P}$, where the second inclusion follows from the closure of $\boldsymbol{\mathcal{P}}$ under composition, tensor products, and adjoints. □

5.2.2 Group-Theoretical- and Π-Quantum Groups

It remains to investigate the (infinitely many) quantum groups induced by the partition categories in part (iv) and (v) of Theorem 3.65. We prove that the $(0,0)$-fragment of their respective intertwiner graphs contain all graphs.

The case for group-theoretical quantum groups follows from a result by Gromada [16]. Let $\mathbf{K}$ be a bilabelled graph and π be a partition of its vertex set. Then we denote by $\mathbf{K}/\pi$ the graph obtained by identifying the blocks of π. Let $\mathbf{G}$ denote the bilabelled graph associated to the partition  . Then the following result holds.

Lemma 5.16 (Lemma 2.31 in [16])**.** *Let $\mathcal{F}$ be a superset of $\langle \mathbf{G} \rangle_{\circ,\otimes,*}$. Then $\mathcal{F}$ is closed under computing quotients $\mathbf{K} \mapsto \mathbf{K}/\pi$, where all unlabelled vertices of $\mathbf{K}$ are singletons in π, i.e. are not identified in $\mathbf{K}/\pi$.*

In particular, we can identify arbitrary labelled vertices. The general idea now is to start with m copies of $\mathbf{A}$ and identify some of their endpoints. This way we can construct all graphs with m edges. However, we need to be slightly careful with this construction to be able to remove any remaining labels afterwards. We achieve this by using double labels.

Lemma 5.17. *Let Q be a graph-theoretic quantum group and $\mathcal{F}$ the set of its intertwiner graphs. Then $\mathcal{F}(0,0)$ is the class of all graphs.*

Proof. We first construct a modified version of $\mathbf{A}$ that has two in- and two out-labels. Note that $\mathbf{M}^{1,3} = (\mathbf{I} \otimes \mathbf{M}^{0,2}) \circ \mathbf{G}$ and $\mathbf{M}^{2,4} = (\mathbf{M}^{1,3} \otimes \mathbf{M}^{1,3}) \circ (\mathbf{I}^{\otimes 2} \otimes \mathbf{M}^{2,0} \otimes \mathbf{I}^{\otimes 2})$. We then let

$$\tilde{\mathbf{A}} := \mathbf{M}^{2,4} \circ \mathbf{A}^{\otimes 4} \circ \mathbf{M}^{4,2}.$$

We now show that we can construct any connected graph, the statement then follows from the closure of $\mathcal{F}(0,0)$ under tensor products.

First note that $K_1 = \mathbf{M}^{0,2}\mathbf{M}^{2,0} \in \mathcal{F}(0,0)$. Suppose then that K is a connected graph with $V(K) = \{v_1, \dots, v_n\}$, $n \geq 2$, and $|E(K)| = m$. We let $\mathbf{K}_0 = \tilde{\mathbf{A}}^{\otimes m}$. Then $\mathbf{K}_0$ has $2m$ in-labels and out-labels respectively, and every vertex is labelled. We identify the m copies of $\tilde{\mathbf{A}}$ with the edges $v_iv_j \in E(K)$ and fix a function $\lambda \colon V(\mathbf{K}_0) \to V(K)$ mapping the vertices of v_iv_j to $\{v_i, v_j\}$. We then define the partition $\pi = \{\{\lambda(w) = v \mid w \in V(\mathbf{K}_0)\} \mid v \in V(K)\}$. Then $\mathbf{K}_0/\pi = (K, \mathbf{l}, \mathbf{l}) =: \mathbf{K}$, where $\mathbf{l}$ is of the form $(w^{v_1}, w^{v_1}, w^{v_1}, w^{v_1}, \dots, w^{v_n}, w^{v_n})$, with an even number of copies of w^{v_i} for all $i \in [n]$ and $|\mathbf{l}| = 2m$. Hence,

$$K = (\mathbf{M}^{0,2})^{\otimes m} \circ \mathbf{K} \circ (\mathbf{M}^{2,0})^{\otimes m}.$$

yields the desired graph. □

Finally, let us turn to the quantum groups $G_n(\Pi_\infty)$ and $G_n(\Pi_k)$. First, we note that $\Pi_1 = \langle$  $\rangle$, which corresponds to the quantum hyperoctahedral group and thus to the class of planar graphs. For $G_n(\Pi_2)$, we follow a similar approach as in Lemma 5.11. We will again start with n isolated vertices that each have two out-labels and then construct gadgets that allow us to add edges between arbitrary vertices. In fact, only the swap gadget will differ slightly.

Lemma 5.18. *Let $\mathcal{F}$ be the class of intertwiner graphs of $G_n(\Pi_2)$. Then $\mathcal{F}(0,0)$ is the class of all graphs.*

Proof. We first show that we have access to $\mathbf{M}^{2,2}$. Let $\mathbf{\Pi}_2$ denote the bilabelled graph associated to π_2. Then we can construct $\mathbf{M}^{4,0}$ as

$$\mathbf{M}^{4,0} = (\mathbf{I}^{\otimes 4} \otimes \mathbf{M}^{0,2} \otimes \mathbf{M}^{0,2}) \circ \mathbf{\Pi}_2.$$

$\mathbf{M}^{2,2}$ is now obtained by "rotating" $\mathbf{M}^{4,0}$: Observe that for any bilabelled graph $(F, (a_1, \dots, a_k), (b_1, \dots, b_\ell)) \in \mathcal{G}(k, \ell)$ we can obtain the graph $(F, (a_1, \dots, a_{k-1}), (b_1, \dots, b_\ell, a_k)) \in \mathcal{G}(k-1, \ell+1)$ by parallel composition with $\mathbf{I}$ and then composing with $\mathbf{I}^{\otimes(k-1)} \otimes \mathbf{M}^{0,2}$. Concretely, we have

$$\mathbf{M}^{2,2} = (\mathbf{I}^{\otimes 2} \otimes \mathbf{M}^{0,2}) \circ \Big(\big((\mathbf{I}^{\otimes 3} \otimes \mathbf{M}^{0,2}) \circ (\mathbf{M}^{4,0} \otimes \mathbf{I}) \big) \otimes \mathbf{I} \Big).$$

We show this procedure in Figure 5.3, where we omit some of the identity graphs to improve legibility. Recall then the construction of $\hat{\mathbf{A}}$ from Lemma 5.11 using $\mathbf{M}^{2,2}$ and $\mathbf{A}$. Then $\hat{\mathbf{A}}$ is an edge whose vertices both have two in-labels and out-labels respectively. It remains to show that we can construct a graph that acts as a swap map. It turns out that we can construct a *fat swap* $\tilde{\mathbf{S}}$ that acts similarly to the $\hat{\mathbf{S}}$ constructed in Lemma 5.11. We again achieve this by rotating $\mathbf{\Pi}_2$, see Figure 5.4.

Now to construct an arbitrary graph $K \in \mathcal{G}$ with $|G| =: n$, we again start with n copies of $\mathbf{M}^{2,0}$ that we identify with the vertices $v_1, \dots, v_n \in V(G)$. Then composition with $\mathbf{I}^{\otimes 2(i-1)} \otimes \hat{\mathbf{A}} \otimes \mathbf{I}^{\otimes 2(n-i-1)}$ adds an edge between v_i and v_{i+1} while preserving the out-labels. Composition with $\tilde{\mathbf{S}}_i := \mathbf{I}^{\otimes 2(i-1)} \otimes \tilde{\mathbf{S}} \otimes \mathbf{I}^{\otimes 2(n-i-1)}$ swaps v_i and v_{i+1} in the sense that for $\mathbf{K} = (K, (v_1, v_1, \dots, v_n, v_n), ())$, we have

$$\tilde{\mathbf{S}}_i \circ \mathbf{K} = (K, (v_1, v_1, \dots, v_{i+1}, v_{i+1}, v_i, v_i, \dots, v_n, v_n), ()).$$

Together, the gadgets $\hat{\mathbf{A}}$ and $\tilde{\mathbf{S}}$ thus allow us to add edges between arbitrary vertices. Finally, composition with $(\mathbf{M}^{0,2})^{\otimes n}$ removes any remaining labels. □

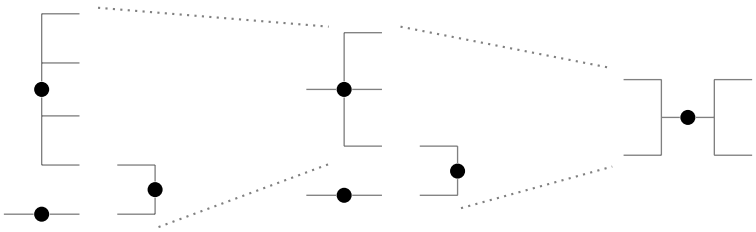


Figure 5.3: Constructing $\mathbf{M}^{2,2}$ from $\mathbf{M}^{0,4}$.

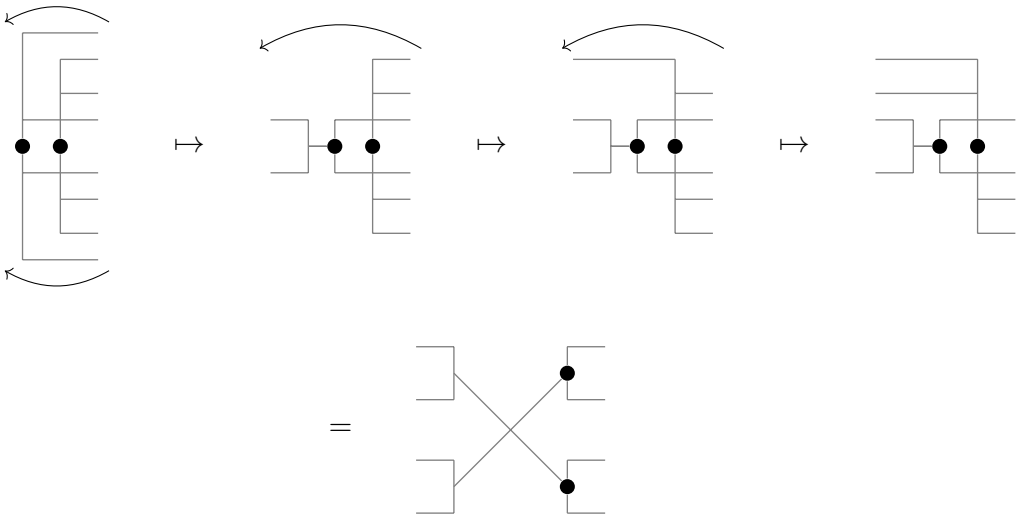


Figure 5.4: Constructing the Fat Swap $\tilde{\mathbf{S}}$ by rotating $\mathbf{\Pi}_2$.

The case for arbitrary $k \geq 2$ now follows using a result on partition categories by Raum and Weber.

Lemma 5.19 (Theorem 3.9 in [26]). *The partition categories Π_k satisfy*

$$\Pi_2 \subsetneq \Pi_3 \subsetneq \cdots \subsetneq \Pi_\infty \subsetneq \langle \; \rangle.$$

Corollary 5.20. *Let $\mathcal{F}$ be the class of intertwiner graphs of $G_n(\Pi_k)$ for $k \geq 2$ and $\mathcal{F}^\infty$ the class of intertwiner graphs of $G_n(\Pi_\infty)$. Then $\mathcal{F}(0,0) = \mathcal{F}^\infty(0,0) = \mathcal{G}$, the class of all graphs.*

Raum and Weber's result moreover yields an alternative, less constructive, proof of Lemma 5.17. We summarize the findings of this section in the following theorem.

Theorem 5.21. *Let Q be an easy quantum group and $\mathcal{F}$ the class of its intertwiner graphs. Then*

(*i*) *$\mathcal{F}(0,0)$ is the class of all graphs if Q is either*

- *the symmetric group S_n,*
- *the hyperoctahedral group H_n,*
- *the modified symmetric group S'_n,*
- *$G_n(E_h)$,*
- *$G_n(E_h^s)$,*
- *a group-theoretical quantum group,*
- *$G_n(\Pi_k)$ for some $k \geq 2$,*
- *$G_n(\Pi_\infty)$.*

(*ii*) *$\mathcal{F}(0,0)$ is the class of all planar graphs if Q is either*

- *the quantum symmetric group S_n^+,*
- *the quantum modified symmetric group $S_n'^+$,*
- *the quantum hyperoctahedral group H_n^+.*

(*iii*) *$\mathcal{F}(0,0)$ is the class of all paths and cycles if Q is either*

- *the bistochastic group B_n,*
- *the quantum bistochastic group B_n^+,*
- *the modified bistochastic group B'_n,*
- *the quantum modified bistochastic group $B_n^{\#+}$,*
- *the quantum complexly modified bistochastic group $B_n'^+$,*
- *$G_n(E'_b)$.*

(*iv*) *$\mathcal{F}(0,0)$ is the class of all cycles if Q is either*

- *the orthogonal group O_n,*
- *the quantum orthogonal group O_n^+,*
- *$G_n(E_o)$.*

Moreover, every Q falls into one of these cases.

5.3 Counting Homomorphisms from Intertwiner Graphs

We now turn to clarifying the connection of intertwiner graph categories, graph instantiations, and homomorphism indistinguishability. Namely, we prove the following theorem.

Theorem 5.22. *Let Q be an easy quantum group with generators $M_1, \dots, M_m$ and let $\mathcal{F}$ be the set of its intertwiner graphs. Let G and H be graphs. Then the following are equivalent.*

(*i*) $G \cong_{\mathcal{F}} H$.

(*ii*) *There exists a weak isomorphism between $Q(G)$ and $Q(H)$.*

(*iii*) *There exists a quantum orthogonal matrix $\mathcal{U}$ with $\mathcal{U}^{\otimes k} M = M \mathcal{U}^{\otimes \ell}$ for all $M \in C^Q(k, \ell)$, such that $\mathcal{U} A_G = A_H \mathcal{U}$.*

(*iv*) *There exists a quantum orthogonal matrix $\mathcal{U}$ with $\mathcal{U}^{\otimes k} M_i = M_i \mathcal{U}^{\otimes \ell}$ for all $i \in [m]$ and $M_i \in C^Q(\ell, k)$, such that $\mathcal{U} A_G = A_H \mathcal{U}$.*

We split the proof of the theorem into two sections. In this section, we examine the connection between homomorphism indistinguishability and the structure of intertwiner spaces through weak isomorphisms. We show that graphs G and H are homomorphism indistinguishable over the unlabelled fragments of the intertwiner graphs of an easy quantum group Q if and only if there exists a weak isomorphism between $Q(G)$ and $Q(H)$, hence proving (i) $\Leftrightarrow$ (ii).

In Section 5.4, we want to show that the existence of a weak isomorphism is equivalent to a statement that more closely resembles the formulation of quantum isomorphism as a C^*-valued linear system of equations $\mathcal{U} A_G = A_H \mathcal{U}$. To that end, we first show (iv) $\Rightarrow$ (ii), proving that any quantum matrix satisfying the intertwiner relations of the underlying easy quantum group induces a weak isomorphism between $Q(G)$ and $Q(H)$. This is proven by generalising the argument outlined in Section 4.4.1. Since the implication (iii) $\Rightarrow$ (iv) is trivial, we conclude by proving that (ii) $\Rightarrow$ (iii) using category-theoretic machinery in the form of a theorem by Neshveyev and Tuset [25].

5.3.1 $\mathcal{F}$-isomorphism implies weak isomorphism of graph instantiations

Suppose then that $G \cong_{\mathcal{F}} H$. We want to show that there exists a weak isomorphism between $Q(G)$ and $Q(H)$. The first requirement is that G and H must be of the same order, which follows without much difficulty.

Observation 5.23. *Let Q be a compact matrix quantum group and $\mathcal{F}$ the class of its intertwiner graphs. Then for all graphs G, H with $G \cong_{\mathcal{F}} H$ it holds that $|G| = |H|$.*

Proof. Note that $\mathcal{F}(0,0)$ contains the single vertex graph K_1. Indeed, we have $K_1 = \mathbf{M}^{0,2}\mathbf{M}^{2,0}$. The result now follows from the fact that $\hom(K_1, G) = |G|$ for all graphs G. □

The central observation is that for any graphs X and Y, C_X^Q and C_Y^Q already look quite similar on the surface, since the intertwiner spaces are generated by the homomorphism tensors of the same bilabelled graphs. In particular, the individual vector spaces $C_X^Q(k, \ell)$ and $C_Y^Q(k, \ell)$ are essentially spanned by the "same" vectors. Now if the scalar product between any two vectors in $C_X^Q(k, \ell)$ always equals that of the corresponding vectors in $C_Y^Q(k, \ell)$, then we can conclude by a Gram-Schmidt argument that there exists a unitary linear map between the two vector spaces that maps a vector in $C_X^Q(k, \ell)$ to its analogue in $C_Y^Q(k, \ell)$. This is formalised in the following lemma by Grohe, Rattan, and Seppelt.

Lemma 5.24 (Lemma 2.1 in [14]). *Let I be a (possibly infinite) index set and $(v_i)_{i\in I}$ and $(w_i)_{i\in I}$ two sequences of vectors in a finite-dimensional vector space such that $\langle v_i, v_j\rangle = \langle w_i, w_j\rangle$ for all $i, j \in I$. Denote by V, W the vector spaces spanned by the v_i, w_i respectively. Then there exists a unitary linear map $\Phi\colon V \to W$ such that $\Phi(v_i) = w_i$ for all $i \in I$.*

The next lemma further implies that the scalar product of vectors in C_G^Q and C_H^Q is precisely given by homomorphism counts of unlabelled intertwiner graphs of Q.

Lemma 5.25. *Let Q be a compact matrix quantum group. For all k, ℓ, there exists a scalar product on $C^Q(k,\ell)$ defined by*

$$\langle T_1, T_2\rangle \coloneqq (M^{0,2})^{\otimes k}(I^{\otimes k} \otimes T_1^* T_2)(M^{2,0})^{\otimes k}$$

for all $T_1, T_2 \in C^Q(k,\ell)$.

Proof. We show that $\langle\cdot,\cdot\rangle$ is indeed a scalar product. Let $T_1, T_2, T_3 \in C^Q(k,\ell)$. It is a straightforward check of dimensions to see that $\langle T_1, T_2\rangle \in \mathbb{C}$, so it remains to show that

(a) $\langle T_1, T_2\rangle^* = \langle T_2, T_1\rangle$

(b) $\langle T_1, aT_2 + bT_3\rangle = a\langle T_1, T_2\rangle + b\langle T_1, T_3\rangle$ for all $a, b \in \mathbb{C}$

(c) $\langle T_1, T_1\rangle \geq 0$

We have

$$\begin{aligned}
\langle T_1, T_2\rangle^* &= \left[(M^{0,2})^{\otimes k}(I^{\otimes k} \otimes T_1^* T_2)(M^{2,0})^{\otimes k}\right]^* \\
&= (M^{0,2})^{\otimes k}(I^{\otimes k} \otimes T_1^* T_2)^*(M^{2,0})^{\otimes k} \\
&= (M^{0,2})^{\otimes k}((I^{\otimes k})^* \otimes (T_1^* T_2)^*)(M^{2,0})^{\otimes k} \\
&= (M^{0,2})^{\otimes k}(I^{\otimes k} \otimes T_2^* T_1)(M^{2,0})^{\otimes k} \\
&= \langle T_2, T_1\rangle,
\end{aligned}$$

while (b) follows from the linearity of matrix multiplication and the tensor product. To see that (c) holds, observe that

$$\begin{aligned}
\langle T, T\rangle &= (M^{0,2})^{\otimes k}(I^{\otimes k} \otimes T^* T)(M^{2,0})^{\otimes k} \\
&= \left[(I^{\otimes k} \otimes T)(M^{2,0})^{\otimes k}\right]^* \left[(I^{\otimes k} \otimes T)(M^{2,0})^{\otimes k}\right] \\
&=: v^* v,
\end{aligned}$$

where $v \in \mathbb{C}^{n^{k+\ell}}$, and thus $v^* v \geq 0$. □

Since the construction only involves the intertwiners $M^{2,0}$ and I, which correspond to bilabelled graphs, we can extend $\langle\cdot,\cdot\rangle$ to (k,ℓ)-bilabelled graphs $\mathbf{K}, \mathbf{K}'$ for all $k, \ell \in \mathbb{N}$. Then $\langle \mathbf{K}, \mathbf{K}'\rangle$ is an unlabelled graph, and we have $\langle \mathbf{K}, \mathbf{K}'\rangle_G = (\langle \mathbf{K}_G, \mathbf{K}'_G\rangle) \in C_G^Q(0,0)$. We can now prove the following lemma.

Lemma 5.26. *Let Q be an easy quantum group, $\mathcal{F}$ the set of its intertwiner graphs, and let G and H be graphs such that $G \cong_{\mathcal{F}} H$. Then there exists a weak isomorphism $\Phi\colon C_G^Q \to C_H^Q$.*

Proof. First note that by Observation 5.23, we have $C_G^Q(k,\ell), C_H^Q(k,\ell) \subseteq \mathbb{C}^{n^k \times n^\ell}$ for all $k, \ell \in \mathbb{N}$. We now define sequences $(v_{\mathbf{K}})_{\mathbf{K}\in\mathcal{F}} \subseteq C_G^Q$, $(w_{\mathbf{K}})_{\mathbf{K}\in\mathcal{F}} \subseteq C_H^Q$ by setting $v_{\mathbf{K}} \coloneqq \mathbf{K}_G$ and $w_{\mathbf{K}} \coloneqq \mathbf{K}_H$.

Then $\{v_{\mathbf{K}} \mid \mathbf{K} \in \mathcal{F}(\ell,k)\}$ spans $C_G^Q(\ell,k)$ and $\{w_{\mathbf{K}} \mid \mathbf{K} \in \mathcal{F}(\ell,k)\}$ spans $C_H^Q(\ell,k)$. Moreover, we have for all $k,\ell \in \mathbb{N}$ and $\mathbf{K},\mathbf{K}' \in \mathcal{F}(\ell,k)$

$$\begin{aligned}\langle v_{\mathbf{K}}, v_{\mathbf{K}'}\rangle &= \mathrm{soe}(\langle \mathbf{K},\mathbf{K}'\rangle_G)\\ &= \hom(\langle \mathbf{K},\mathbf{K}'\rangle, G)\\ &= \hom(\langle \mathbf{K},\mathbf{K}'\rangle, H)\\ &= \mathrm{soe}(\langle \mathbf{K},\mathbf{K}'\rangle_H)\\ &= \langle w_{\mathbf{K}}, w_{\mathbf{K}'}\rangle,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product defined in Lemma 5.25. Consequently, by Lemma 5.24 there exists for all $\ell,k \in \mathbb{N}$ a unitary linear map $\Phi_{\ell,k}\colon C_G^Q(\ell,k) \to C_H^Q(\ell,k)$, mapping $v_{\mathbf{K}}$ to $w_{\mathbf{K}}$ for all $\mathbf{K} \in \mathcal{F}$.

Claim 1. *$\Phi := \bigcup_{k,\ell\in\mathbb{N}} \Phi_{\ell,k}$ is a weak isomorphism $C_G^Q \to C_H^Q$.*

Proof. By linearity, it suffices to show that for all $\mathbf{K}_G, \mathbf{K}'_G \in C_G^Q$ with $\mathbf{K},\mathbf{K}' \in \mathcal{F}$ it holds that

(a) $\Phi(A_G) = A_H$ and $\Phi(M_i) = M_i$ for all $i \in [k]$,

(b) $\Phi(\mathbf{K}_G\mathbf{K}'_G) = \Phi(\mathbf{K}_G)\Phi(\mathbf{K}'_G)$ if $\mathbf{K}_G, \mathbf{K}'_G$ are compatible,

(c) $\Phi(\mathbf{K}_G \otimes \mathbf{K}'_G) = \Phi(\mathbf{K}_G) \otimes \Phi(\mathbf{K}'_G)$,

(d) $\Phi(\mathbf{K}_G^*) = \Phi(\mathbf{K}_G)^*$,

(e) For all $\ell,k \in \mathbb{N}$, the restriction of Φ to $C_G^Q(\ell,k)$ is a bijective linear map to $C_H^Q(\ell,k)$.

First note that the restriction of Φ to $C_G^Q(\ell,k)$ is just $\Phi_{\ell,k}$, which is a unitary linear map and thus bijective. Moreover, we have $A_G = \mathbf{A}_G = v_A$ and $A_H = \mathbf{A}_H = w_A$ and thus $\Phi(A_G) = A_H$; as well as $M_i = v_{\mathbf{M}_i} = w_{\mathbf{M}_i}$ (since $\mathbf{M}_i$ is edgeless and G, H have the same order) and hence $\Phi(M_i) = M_i$. To see that (b) holds, observe that for compatible $\mathbf{K}_G, \mathbf{K}'_G \in C_G^Q$,

$$\begin{aligned}\Phi(\mathbf{K}_G\mathbf{K}'_G) = \Phi((\mathbf{K}\mathbf{K}')_G) = \Phi(v_{\mathbf{K}\mathbf{K}'}) = w_{\mathbf{K}\mathbf{K}'} = (\mathbf{K}\mathbf{K}')_H = \mathbf{K}_H\mathbf{K}'_H = w_{\mathbf{K}}w_{\mathbf{K}'}\\ = \Phi(\mathbf{K}_G)\Phi(\mathbf{K}'_G).\end{aligned}$$

The proof of (c) is analogous. Finally, we have

$$\Phi(\mathbf{K}_G^*) = \Phi((\mathbf{K}^*)_G) = \Phi(v_{\mathbf{K}^*}) = w_{\mathbf{K}^*} = (\mathbf{K}^*)_H = \mathbf{K}_H^* = \Phi(\mathbf{K}_G)^*,$$

so (d) is also satisfied, and Φ is a weak isomorphism. ■

This concludes the proof. □

5.3.2 Weak isomorphism of graph instantiations implies $\mathcal{F}$-isomorphism

The converse direction is a generalisation of the argument in Section 4.4.1. The main observation is again that a weak isomorphism mirrors the construction of intertwiners using composition, tensor product, and adjoints. In particular, the following holds.

Lemma 5.27. *Let Q be an easy quantum group, $\mathcal{F}$ the set of its intertwiner graphs, and G, H be graphs such that there exists a weak isomorphism $\Phi\colon C_G^Q \to C_H^Q$. Then for all $\mathbf{K} \in \mathcal{F}$ it holds that $\Phi(\mathbf{K}_G) = \mathbf{K}_H$.*

Proof. Since $\mathbf{K} \in \mathcal{F}$, there exists a construction of $\mathbf{K}$ from generators $\mathbf{M}_1, \dots, \mathbf{M}_k \in \mathcal{F}$ and $\mathbf{A}$ using the operations $\circ, \otimes, *$. We fix such a construction and proceed inductively. Per definition of weak isomorphisms, we have $\Phi((\mathbf{M}_i)_G) = \Phi(M_i) = M_i = (\mathbf{M}_i)_H$ for all $i \in [k]$ and $\Phi(\mathbf{A}_G) = \Phi(A_G) = A_H = \mathbf{A}_H$. Now suppose $\mathbf{K} = \mathbf{K}_1 \circ \mathbf{K}_2$. Then

$$\Phi(\mathbf{K}_G) = \Phi((\mathbf{K}_1)_G(\mathbf{K}_2)_G) = \Phi((\mathbf{K}_1)_G)\Phi((\mathbf{K}_2)_G) = (\mathbf{K}_1)_H(\mathbf{K}_2)_H = (\mathbf{K}_1\mathbf{K}_2)_H = \mathbf{K}_H.$$

The cases $\mathbf{K} = \mathbf{K}_1 \otimes \mathbf{K}_2$ and $\mathbf{K} = \mathbf{K}_1^*$ are analogous. □

Together with the results on bilabelled graphs and homomorphism tensors of Chapter 4, this allow us to prove implication (ii) ⇒ (i) of Theorem 5.22.

Lemma 5.28. *Let Q be an easy quantum group, $\mathcal{F}$ the set of its intertwiner graphs, and G and H be graphs that admit a weak isomorphism Φ between $Q(G)$ and $Q(H)$. Then $G \cong_{\mathcal{F}} H$.*

Proof. By Lemma 5.6, the weak isomorphism $\Phi\colon C_G^Q \to C_H^Q$ is the identity on $C_G^Q(0,0)$. Now let $\mathbf{F} = (F, (), ()) \in \mathcal{F}(0,0)$ be arbitrary. Then we have $\mathbf{F}_G \in C_G^Q(0,0)$ and thus

$$\hom(F, G) = \operatorname{soe}(\mathbf{F}_G) = \operatorname{soe}(\Phi(\mathbf{F}_G)) = \operatorname{soe}(\mathbf{F}_H) = \hom(F, H).$$ □

5.4 Monoidal Equivalence and intertwining Quantum Matrices

So far we have seen that the characterisation of intertwiners as homomorphism tensors implies that homomorphism-indistinguishability of graphs G and H over the unlabelled intertwiner graphs of an easy quantum group Q means that the intertwiner spaces of the graph instantiations $Q(G)$ and $Q(H)$ "look the same". While this is already an interesting result, it would be great to obtain a characterisation of G and H in terms of a C*-algebra valued linear system, analogous to quantum isomorphism. Such a characterisation would fit well into the existing landscape of homomorphism indistinguishability relations, which often enjoy similar descriptions. In this section we derive such a characterisation, proving the equivalences (ii) ⇔ (iii) ⇔ (iv) of Theorem 5.22.

5.4.1 Quantum Matrices induce Weak Isomorphisms

The first direction is a straightforward generalisation of Mančinska and Roberson's proof of Lemma 4.38.

Lemma 5.29. *Let Q be an easy quantum group with generators $M_1, \dots, M_m$ and let G and H be graphs. If there exists a quantum orthogonal matrix $\mathcal{U}$ with $\mathcal{U}A_G = A_H\mathcal{U}$ such that $\mathcal{U}^{\otimes k}M_i = M_i\mathcal{U}^{\otimes \ell}$ for all $i \in [m]$ and $M_i \in C^Q(k,\ell)$, then there exists a weak isomorphism $\Phi\colon C_G^Q \to C_H^Q$ between $Q(G)$ and $Q(H)$.*

Proof. We show that the map $\Phi\colon C_G^Q \to C_H^Q$ given by $\mathcal{U}^{\otimes \ell}T = \Phi(T)\mathcal{U}^{\otimes k}$ for $T \in C_G^Q(\ell, k)$ for all $k, \ell \in \mathbb{N}$ is a weak isomorphism. To see that Φ is well-defined, note that $T\mathcal{U}^{\otimes k} = T'\mathcal{U}^{\otimes k}$ implies that $T = T'$, since $\mathcal{U}^{\otimes k}$ is orthogonal and thus invertible.

Now consider the generators $M_1, \dots, M_k$ of Q. Per assumption $\mathcal{U}^{\otimes \ell}M_i = M_i\mathcal{U}^{\otimes k}$ for all $i \in [k]$ with $M_i \in C^Q(\ell, k)$ and $\mathcal{U}A_G = A_H\mathcal{U}$, so we have $\Phi(M_i) = M_i$ and $\Phi(A_G) = A_H$. Since the M_i and A_G generate C_G^Q, to prove that Φ is defined everywhere on C_G^Q, it suffices to show that Φ is linear and commutes with tensor product, composition and hermitian adjoint.

Suppose that $\mathcal{U}^{\otimes \ell}T_i = T_i'\mathcal{U}^{\otimes k}$ and $\lambda_i \in \mathbb{C}$ for $i \in [m]$. Then it is straightforward to see that

$$\mathcal{U}^{\otimes \ell}\left(\sum_{i=1}^m \lambda_i T_i\right) = \left(\sum_{i=1}^m \lambda_i T_i'\right)\mathcal{U}^{\otimes k}.$$

Now let $T_1, T_1' \in C_G^Q(\ell, k), T_2, T_2' \in C_G^Q(k, m)$ with $\mathcal{U}^{\otimes \ell} T_1 = T_1' \mathcal{U}^{\otimes k}$ and $\mathcal{U}^{\otimes k} T_2 = T_2' \mathcal{U}^{\otimes m}$. Then

$$\mathcal{U}^{\otimes \ell}(T_1 T_2) = T_1' \mathcal{U}^{\otimes k} T_2 = T_1' T_2' \mathcal{U}^{\otimes m}.$$

Next let $T_1, T_1' \in C_G^Q(\ell, k), T_2, T_2' \in C_G^Q(r, s)$ with $\mathcal{U}^{\otimes \ell} T_1 = T_1' \mathcal{U}^{\otimes k}$ and $\mathcal{U}^{\otimes r} T_2 = T_2' \mathcal{U}^{\otimes s}$. Then

$$\begin{aligned}\mathcal{U}^{\otimes(\ell+r)}(T_1 \otimes T_2) &= (\mathcal{U}^{\otimes \ell} \otimes \mathcal{U}^{\otimes r})(T_1 \otimes T_2) = \mathcal{U}^{\otimes \ell} T_1 \otimes \mathcal{U}^{\otimes r} T_2 = T_1' \mathcal{U}^{\otimes k} \otimes T_2' \mathcal{U}^{\otimes s} \\ &= (T_1' \otimes T_2') \mathcal{U}^{\otimes(k+s)}\end{aligned}$$

Finally, let $T, T' \in C_G^Q(\ell, k)$ with $\mathcal{U}^{\otimes \ell} T = T' \mathcal{U}^{\otimes k}$. Then

$$\begin{aligned}\mathcal{U}^{\otimes k} T^* &= \mathcal{U}^{\otimes k}[T^*(\mathcal{U}^{\otimes \ell})^*]\mathcal{U}^{\otimes \ell} = \mathcal{U}^{\otimes k}[\mathcal{U}^{\otimes \ell} T]^* \mathcal{U}^{\otimes \ell} = \mathcal{U}^{\otimes k}[T' \mathcal{U}^{\otimes k}]^* \mathcal{U}^{\otimes \ell} = \mathcal{U}^{\otimes k}(\mathcal{U}^{\otimes k})^* T'^* \mathcal{U}^{\otimes \ell} \\ &= T'^* \mathcal{U}^{\otimes \ell}.\end{aligned}$$

We thus have shown that Φ is linear and commutes with the necessary operations. Since $M_1, \ldots, M_k, A_H$ generate C_H^Q and these intertwiners are in the image of Φ, it follows that Φ is surjective. To see that Φ is injective, suppose that $\Phi(T) = \Phi(T')$. Then $\mathcal{U}^{\otimes k} T = \Phi(T) \mathcal{U}^{\otimes \ell} = \Phi(T') \mathcal{U}^{\otimes \ell} = \mathcal{U}^{\otimes k} T'$, thus the orthogonality of $\mathcal{U}^{\otimes k}$ implies that $T = T'$. It follows that Φ is a weak isomorphism, which completes the proof. □

5.4.2 A brief excursion into Category Theory

In order to prove the converse direction, we need to introduce a few more concepts from category theory. In the following let Q be a compact matrix quantum group and G, H be graphs. Our first goal is to show that a weak isomorphism between $Q(G)$ and $Q(H)$ is really just a *monoidal equivalence* of $Q(G)$ and $Q(H)$ in disguise.

Recall the definition of the representation category $\mathrm{Rep}(Q)$ from Section 3.4.1. It turns out that $\mathrm{Rep}(Q)$ forms a so called *C*-tensor category*. For a full definition of C*-tensor categories see Definition 2.3.13 in [20][1]—for our purposes it suffices to know that they are monoidal categories in the sense of Remark 3.58 that allow linear combinations of morphisms.

Definition 5.30. Let $\mathcal{C}, \mathcal{C}'$ be C*-tensor categories. A *tensor functor* $\mathcal{C} \to \mathcal{C}'$ is a functor $F \colon \mathcal{C} \to \mathcal{C}'$ that is linear on morphisms, together with natural isomorphisms

$$F_2 \colon F(U) \otimes F(V) \to F(U \otimes V)$$

such that the diagram

$$\begin{array}{ccccc}(F(U) \otimes F(V)) \otimes F(W) & \xrightarrow{F_2 \otimes \mathrm{id}} & F(U \otimes V) \otimes F(W) & \xrightarrow{F_2} & F((U \otimes V) \otimes W) \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\ F(U) \otimes (F(V) \otimes F(W)) & \xrightarrow{\mathrm{id} \otimes F_2} & F(U) \otimes F(V \otimes W) & \xrightarrow{F_2} & F(U \otimes (V \otimes W))\end{array}$$

commutes, and $F(\mathbf{1}) \cong \mathbf{1}'$. We call F *unitary* if additionally $F(g^*) = F(g)^*$ for all morphisms g and F_2 is unitary.

Definition 5.31. We call two compact quantum groups Q_1 and Q_2 (*unitarily*) *monoidally equivalent* if there exists a (unitary) tensor functor $\mathrm{Rep}(Q_1) \to \mathrm{Rep}(Q_2)$ that is fully faithful and essentially surjective.

[1] For simplicity and ease of notation, we tacitly assume that every C*-tensor category is *strict*, i.e. that the associator α as well as the left and right unitors λ and ρ are the identity. This does not change the validity of the results, since $\mathrm{Rep}(Q)$ is strict for all compact quantum groups Q.

Now suppose we have a weak isomorphism $\Phi\colon C_G^Q \to C_H^Q$. Recall that the sets C_G^Q and C_H^Q are the morphisms of the reduced representation categories $\mathrm{Rep}_0(Q(G))$ and $\mathrm{Rep}_0(Q(H))$ respectively. The objects of these categories are the tensor powers of the fundamental representations $\mathcal{U}_G$ of $Q(G)$ and $\mathcal{U}_H$ of $Q(H)$, so given Φ we can construct a functor $F\colon \mathrm{Rep}_0(Q(G)) \to \mathrm{Rep}_0(Q(H))$ with $F(\mathcal{U}_G^{\otimes k}) = \mathcal{U}_H^{\otimes k}$ and $F(T) = \Phi(T)$ for all $T \in \mathrm{Mor}(\mathcal{U}_G^{\otimes k}, \mathcal{U}_G^{\otimes \ell})$ and $k, \ell \in \mathbb{N}$.[2]

We will show that we can extend this functor from Rep_0 to the full category Rep. The main observation is that for any compact matrix quantum group Q, the category $\mathrm{Rep}(Q)$ is the pseudoabelian completion of $\mathrm{Rep}_0(Q)$. Roughly speaking, in a pseudoabelian category every idempotent endomorphism decomposes into the direct sum of kernel and image. Now for compact matrix quantum groups, every finite-dimensional representation is a direct summand of a tensor power of the fundamental representation and the projectors onto the corresponding subspaces are exactly the idempotent endomorphisms. Consequently, moving to the pseudoabelian completion of the reduced representation category recovers the other finite-dimensional representations as the images of the projectors.

Definition 5.32. The *pseudoabelian completion* of a category $\mathcal{C}$ is defined as the category given by

Objects:	(A, e), $A \in \mathrm{Obj}(\mathcal{C})$, $e \in \mathrm{Mor}(A, A)$ idempotent,
Morphisms:	$\mathrm{Mor}((A, e), (B, f)) = f\,\mathrm{Mor}(A, B)e$
Composition:	induced by the composition in $\mathcal{C}$.

Intuitively, the object (A, e) corresponds to the "subspace" of A that e projects onto. Concretely, this means that for $Q = (C(Q), \mathcal{U}, n)$, we can assume $\mathrm{Rep}(Q)$ to be of the following form.

Objects:	$(\mathcal{U}^{\otimes k}, T)$, $T\colon (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k}$ linear idempotent, $\mathcal{U}^{\otimes k}T = T\mathcal{U}^{\otimes k}$, $k \in \mathbb{N}$,
Morphisms:	$\mathrm{Mor}((\mathcal{U}^{\otimes k}, T), (\mathcal{U}^{\otimes \ell}, T')) = T'ST$, $S \in \mathrm{Mor}(\mathcal{U}^{\otimes k}, \mathcal{U}^{\otimes \ell}) = C(k, \ell)$
Composition:	natural composition.

This allows us to prove the following lemma.

Lemma 5.33. *Let Q be a compact matrix quantum group with generators $M_i \in \mathrm{Mor}((\mathcal{U}_G^{\otimes k_i}, I^{\otimes k_i}), (\mathcal{U}^{\otimes \ell_i}, I^{\otimes \ell_i}))$, $m, k_i, \ell_i \in \mathbb{N}, i \in [m]$, and let G, H be graphs that admit a weak isomorphism Φ between $Q(G)$ and $Q(H)$. Then there exists a unitary tensor functor $F\colon \mathrm{Rep}(Q(G)) \to \mathrm{Rep}(Q(H))$ that is fully faithful and essentially surjective, such that*

(*i*) $F(A_G) = A_H$,

(*ii*) $F(M_i) = M_i$ *for all* $i \in [m]$,

(*iii*) $F(\mathcal{U}_G^{\otimes k}, I^{\otimes k}) = (\mathcal{U}_H^{\otimes k}, I^{\otimes k})$,

(*iv*) F_2 *is the identity.*

In particular, $Q(G)$ and $Q(H)$ are unitarily monoidally equivalent.

Proof. We construct a functor $F\colon \mathrm{Rep}(Q(G)) \to \mathrm{Rep}(Q(H))$ as follows.

On objects:	$(\mathcal{U}_G^{\otimes k}, T) \mapsto (\mathcal{U}_H^{\otimes k}, \Phi(T))$,
On morphisms:	$T'ST \mapsto \Phi(T')\Phi(S)\Phi(T)$, for $T'ST \in \mathrm{Mor}((\mathcal{U}_G^{\otimes k}, T), (\mathcal{U}_G^{\otimes \ell}, T'))$.

To prove that this is well defined, we need to show that $T \in \mathrm{Mor}_{\mathrm{Rep}_0(Q(G))}(\mathcal{U}^{\otimes k}, \mathcal{U}^{\otimes k})$ is idempotent if and only if $\Phi(T)$ is idempotent. If T is idempotent, we have $\Phi(T)\Phi(T) = \Phi(TT) = \Phi(T)$, so $\Phi(T)$ is idempotent. For the converse suppose that $\Phi(T)$ was idempotent, but T was not. Then $\Phi(TT) = \Phi(T)\Phi(T) = \Phi(T)$ but $TT \neq T$, contradicting the injectivity of Φ.

[2] It may be enlightening to verify that that the properties of the weak isomorphism Φ guarantee that F is indeed a valid functor.

To see that F is a valid functor $\mathrm{Rep}(Q(G)) \to \mathrm{Rep}(Q(H))$, we need to show that $F(\mathrm{id}_U) = \mathrm{id}_{F(U)}$ for all objects U and that $F(g \circ f) = F(g) \circ F(f)$ for all morphisms g, h. The identity of $U = (\mathcal{U}_G^{\otimes k}, T)$ is T, so we have $F(T) = \Phi(T) = \mathrm{id}_{F(U)}$. Moreover, for morphisms $T_2 S T_1 \in \mathrm{Mor}((\mathcal{U}^{\otimes k_1}, T_1), (\mathcal{U}^{\otimes k_2}, T_2))$, $T_3 S' T_2 \in \mathrm{Mor}((\mathcal{U}^{\otimes k_2}, T_2), (\mathcal{U}^{\otimes k_3}, T_3))$ we have

$$\begin{aligned} F(T_2 S T_1 \circ T_3 S' T_2) &= \Phi(T_2 S T_1 \circ T_3 S' T_2) \\ &= \Phi(T_2 S T_1) \circ \Phi(T_3 S' T_2) \\ &= \Phi(T_2)\Phi(S)\Phi(T_1) \circ \Phi(T_3)\Phi(S')\Phi(T_2) \\ &= F(T_2 S T_1) \circ F(T_3 S' T_2) \end{aligned}$$

Claim 1. *F is a unitary tensor functor.*

Proof. It is not hard to see that F is linear on morphisms. This follows almost immediately from the linearity of Φ and the fact that it commutes with composition. Moreover, the identities at each object indeed form a natural isomorphism $F_2 \colon F(U) \otimes F(V) \to F(U \otimes V)$: For objects $(\mathcal{U}_G^{\otimes k}, T)$ and $(\mathcal{U}_G^{\otimes \ell}, T')$, we have

$$\begin{aligned} F(\mathcal{U}_G^{\otimes k}, T) \otimes F(\mathcal{U}_G^{\otimes \ell}, T') &= (\mathcal{U}_H^{\otimes k}, \Phi(T)) \otimes (\mathcal{U}_H^{\otimes \ell}, \Phi(T')) \\ &= (\mathcal{U}_H^{\otimes k} \otimes \mathcal{U}_H^{\otimes \ell}, \Phi(T) \otimes \Phi(T')) \\ &= (\mathcal{U}_H^{\otimes (k+\ell)}, \Phi(T \otimes T')) \\ &= F(\mathcal{U}_G^{\otimes (k+\ell)}, T \otimes T') \\ &= F(\mathcal{U}_G^{\otimes k} \otimes \mathcal{U}_G^{\otimes \ell}, T \otimes T'). \end{aligned}$$

where the second equality uses the natural tensor structure on $\mathrm{Rep}(Q(H))$. For morphisms $T_2 S T_1 \in \mathrm{Mor}((\mathcal{U}^{\otimes k_1}, T_1), (\mathcal{U}^{\otimes k_2}, T_2))$, $T_4 S' T_3 \in \mathrm{Mor}((\mathcal{U}^{\otimes k_3}, T_2), (\mathcal{U}^{\otimes k_4}, T_3))$, we have

$$F(T_2 S T_1 \otimes T_4 S' T_3) = \Phi(T_2 S T_1 \otimes T_4 S' T_3) = \Phi(T_2 S T_1) \otimes \Phi(T_4 S' T_3) = F(T_2 S T_1) \otimes F(T_4 S' T_3).$$

The unit $\mathbf{1}_G$ is the object $(\mathcal{U}_G^{\otimes 0}, I^{\otimes 0})$ which, by Lemma 5.6, gets mapped to $F(\mathcal{U}_G^{\otimes 0}, I^{\otimes 0}) = (\mathcal{U}_H^{\otimes 0}, \Phi(I^{\otimes 0})) = (\mathcal{U}_H^{\otimes 0}, I^{\otimes 0})$, which is the unit $\mathbf{1}_H$ in $\mathrm{Rep}(Q(H))$. In particular, $F(\mathbf{1}_G) \cong \mathbf{1}_H$. Finally, the unitarity of F follows immediately from the fact that Φ commutes with the hermitian adjoint. ■

Claim 2. *F is fully faithful and essentially surjective.*

Proof. We have already seen that Φ is bijective on the idempotent intertwiners. This implies that F is essentially surjective, i.e. surjective on objects. Indeed, suppose $(\mathcal{U}_H^{\otimes k}, T) \in \mathrm{Rep}(Q(H))$. Since $T \in C_G^Q(k,k)$ is idempotent, there must exist an idempotent $T' \in C_H^Q(k,k)$ such that $\Phi(T') = T$, and thus $F(\mathcal{U}_G^{\otimes k}, T') = (\mathcal{U}_H^{\otimes k}, \Phi(T')) = (\mathcal{U}_H^{\otimes k}, T)$.

That F is fully faithful, i.e. bijective on morphisms, follows immediately from the fact that the restrictions of Φ to $C_G^Q(k,\ell)$ are bijective for all $k, \ell \in \mathbb{N}$. ■

We have thus shown that there exists a unitary tensor functor $F \colon \mathrm{Rep}(Q(G)) \to \mathrm{Rep}(Q(H))$ that is fully faithful and essentially surjective. Per definition, it follows that $Q(G)$ and $Q(H)$ are unitarily monoidally equivalent. Conditions (i), (ii), (iii) follow immediately from the construction of F. □

In this sense, our weak isomorphisms serve as a combinatorial view on the category-theoretic structure of representation categories. In Remark 3.58, we mentioned that concrete tensor categories with duals such as C_G^Q and C_H^Q induce C*-tensor categories *with duals*. To later state a central result, we will need to make this notion more precise.

Definition 5.34. Let $\mathcal{C}$ be a C*-tensor category. An object $\bar{U}$ is called the *dual* of an object $U \in \mathcal{C}$ if there exists morphisms $R\colon \mathbf{1} \to \bar{U} \otimes U$ and $\bar{R}\colon \mathbf{1} \to U \otimes \bar{U}$ such that

$$U \xrightarrow{\mathrm{id}\otimes R} U \otimes \bar{U} \otimes U \xrightarrow{\bar{R}^*\otimes \mathrm{id}} U \quad \text{and} \quad \bar{U} \xrightarrow{\mathrm{id}\otimes \bar{R}} \bar{U} \otimes U \otimes \bar{U} \xrightarrow{R^*\otimes \mathrm{id}} \bar{U}$$

are the identity morphisms. We call these conditions the *conjugate equations*.

Definition 5.35. A *C*-tensor category with duals* is a C*-tensor category where every element has a dual.

An important result is that for every compact quantum group Q, its representation category $\mathrm{Rep}(Q)$ is a C*-tensor category with duals—see for instance [25, Example 2.2.3].

Finally, we need to introduce the concept of a *fiber functor*. Essentially, this is just a functor mapping representations to their underlying spaces. We give a definition for the case of C*-tensor categories with duals.

Definition 5.36. Let $\mathcal{C}$ be a C*-tensor category with duals. A *fiber functor* is a tensor functor $F\colon \mathcal{C} \to \mathrm{fdHilb}$, where fdHilb denotes the category of finite-dimensional Hilbert spaces.

The representation category $\mathrm{Rep}(Q)$ of any compact quantum group Q admits a canonical fiber functor $F_Q\colon \mathrm{Rep}(Q) \to \mathrm{fdHilb}$ defined as follows.

On objects:	$\mathcal{V} \mapsto \mathbb{C}^m$ for all representations $\mathcal{V} \in C(Q)^{m\times m}$.
On morphisms:	$T \mapsto T$ for all intertwiners T of Q.

5.4.3 Weak Isomorphisms induce Quantum Matrices

We have seen that a weak isomorphism between two graph instatiations $Q(G)$ and $Q(H)$ witnesses that $Q(G)$ and $Q(H)$ are monoidally equivalent. The central result that will allow us to prove the converse of Lemma 5.29 is that any two monoidally equivalent C*-tensor categories admit a *linking algebra* that is compatible with the morphisms in a specific sense. This linking algebra will provide the entries for our desired quantum matrix.

To state this result, we first note the following, which is straightforward to verify.

Lemma 5.37. *Let $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ be C*-tensor categories and let $E\colon \mathcal{C} \to \mathcal{C}'$ and $F\colon \mathcal{C}' \to \mathcal{C}''$ be (unitary) tensor functors. Then $F \circ E\colon \mathcal{C} \to \mathcal{C}''$ is also a (unitary) tensor functor.*

Corollary 5.38. *Let Q be an easy quantum group and G, H be graphs. Suppose $Q(G)$ and $Q(H)$ are monoidally equivalent, witnessed by a tensor functor $F\colon \mathrm{Rep}(Q(G)) \to \mathrm{Rep}(Q(H))$. Then $F_{Q(H)} \circ F$ is a fiber functor $\mathrm{Rep}(Q(G)) \to \mathrm{fdHilb}$.*

Consequently, the monoidal equivalence of $Q(G)$ and $Q(H)$ yields another fiber functor $\mathrm{Rep}(Q(G)) \to \mathrm{fdHilb}$ in addition to the canonical fiber functor $F_{Q(G)}$. This allows us to use the following theorem by Neshveyev and Tuset, which is stated in terms of fiber functors.

Theorem 5.39 (Theorem 2.3.11 in [25])**.** *Let $\mathcal{C}$ be a C*-tensor category with duals and $E, F\colon \mathcal{C} \to$ fdHilb be two fiber functors. Then there exists a C*-algebra $\mathcal{Z}$, and for every object $U \in \mathcal{C}$ a unitary element $X^U \in \mathcal{Z} \otimes B(E(U), F(U))$ such that*

(i) If $T \in \mathrm{Mor}(U, V)$, then $(I \otimes F(T))X^U = X^V(I \otimes E(T))$.

(ii) $(I \otimes F_2)X^U_{12}X^V_{13} = X^{U\otimes V}(I \otimes E_2)$.

(iii) If $(R, \bar{R})$ is a solution of the conjugate equations for U and $\bar{U}$, then

$$(X^U_{12})^*(\mathbf{1} \otimes F_2^*F(\bar{R})) = X^{\bar{U}}_{13}(\mathbf{1} \otimes E_2^*E(\bar{R})).$$

Here $B(E(U), F(U))$ denotes the set of bounded linear operators from $E(U)$ to $F(U)$. The expression X_{12} uses the so called leg numbering notation, which is used to "pad" operators such that they only act on certain subspaces: If $X = \sum_i z_i \otimes T_i \in \mathcal{Z} \otimes B(E(U), F(U))$, then $X_{12} = \sum_i z_i \otimes T_i \otimes I$ and $X_{13} = \sum_i z_i \otimes I \otimes T_i$.

Now if there exists a weak isomorphism between $Q(G)$ and $Q(H)$, Lemma 5.33 guarantees the existence of a tensor functor $\mathrm{Rep}(Q(G)) \to \mathrm{Rep}(Q(H))$, which in turn, as we have already noted, yields two fiber functors $\mathrm{Rep}(Q(G)) \to \mathrm{fdHilb}$. Theorem 5.39 thus already looks quite similar to what we want to prove. To bring it into the form that we need, we first note that we already know the solutions to the conjugate equations for $(\mathcal{U}, I) \in \mathrm{Rep}(Q(G))$.

Lemma 5.40. *Let $(C(Q), \mathcal{U}, n)$ be a compact matrix quantum group. Then $(\mathcal{U}, I)$ is self-dual in $\mathrm{Rep}(Q)$ and $R = \bar{R} = M^{2,0}$ solves the conjugate equations.*

Proof. First note that $M^{2,0}$ is a $(2,0)$-intertwiner, thus in particular a morphism $\mathbf{1} \to (\mathcal{U}, I) \otimes (\mathcal{U}, I)$ in $\mathrm{Rep}(Q)$. The identity at $(\mathcal{U}, I)$ is I, so we have to show that $((M^{2,0})^* \otimes I)(I \otimes M^{2,0}) = I$. Indeed we have

$$\begin{aligned}((M^{2,0})^* \otimes I)(I \otimes M^{2,0}) &= (\sum_i e_i^\mathsf{T} \otimes e_i^\mathsf{T} \otimes I)(\sum_i I \otimes e_i \otimes e_i) = \sum_{ij} e_i^\mathsf{T} \otimes e_i^\mathsf{T} e_j \otimes e_j \\ &= \sum_{ij} e_i^\mathsf{T} \otimes \delta_{ij} \otimes e_j = \sum_i e_i^\mathsf{T} \otimes e_i = I.\end{aligned}$$

□

This allows us to prove the following theorem. For simplicity, we will identify $(\mathcal{U}_G^{\otimes k}, I^{\otimes k}) \in \mathrm{Rep}(Q(G))$ with $\mathcal{U}_G^{\otimes k}$. We can do this without loss of generality, since the subcategory of objects $(\mathcal{U}_G^{\otimes k}, I^{\otimes k})$ is just (isomorphic to) the reduced representation category $\mathrm{Rep}_0(Q(G))$.

Theorem 5.41. *Let $(C(Q), \mathcal{U}, n)$ be an easy quantum group and G and H be graphs of size n that admit a weak isomorphism Φ between $Q(G)$ and $Q(H)$. Then there exists a C*-algebra $\mathcal{Z}$ and for all $\mathcal{U}_G^{\otimes k} \in \mathrm{Rep}(Q(G))$ a unitary matrix $Z^{(k)} \in \mathcal{Z}^{n^k \times n^k}$, such that for all $k, \ell \in \mathbb{N}$*

(*i*) *If $T \in \mathrm{Mor}(\mathcal{U}_G^{\otimes k}, \mathcal{U}_G^{\otimes \ell})$, then $Z^{(\ell)} T = \Phi(T) Z^{(k)}$.*

(*ii*) $Z^{(k)} \otimes Z^{(\ell)} = Z^{(k+\ell)}$.

Moreover, the entries z_{ij} of $Z^{(1)}$ satisfy $z_{ij}^* = z_{ij}$.

Proof. We start with some observations. The existence of a weak isomorphism Φ between $Q(G)$ and $Q(H)$ implies by Lemma 5.33 that there exists a unitary tensor functor $F^{H,G} \colon \mathrm{Rep}(Q(G)) \to \mathrm{Rep}(Q(H))$ such that

(a) $F^{H,G}(A_G) = A_H$,

(b) $F^{H,G}(M_i) = M_i = \Phi(M_i)$ for all generators M_i of Q,

(c) $F_2^{H,G}$ is the identity,

(d) $F^{H,G}(\mathcal{U}_G^{\otimes k}, I) = (\mathcal{U}_H^{\otimes k}, I)$.

By Corollary 5.38, there exist two fiber functors $E, F \colon \mathrm{Rep}(Q(G)) \to \mathrm{fdHilb}$ given by $E = F_{Q(G)}$ and $F = F_{Q(H)} \circ F^{H,G}$. By property (d), we have $E(\mathcal{U}_G^{\otimes k}, I) = F(\mathcal{U}_G^{\otimes k}, I) = (\mathbb{C}^n)^{\otimes k} \cong \mathbb{C}^{n^k}$. This means that we can identify $\mathcal{Z} \otimes B(E(\mathcal{U}_G^{\otimes k}), F(\mathcal{U}_G^{\otimes k})) = \mathcal{Z} \otimes B(\mathbb{C}^{n^k}, \mathbb{C}^{n^k})$ with the $n^k \times n^k$ matrices over $\mathcal{Z}$. To see this, recall that there is a one-to-one correspondence between matrices and the bounded linear operators over finite dimensional Hilbert spaces. Let e_{ij} denote the canonical basis of $B(\mathbb{C}^{n^k}, \mathbb{C}^{n^k})$, that is, the matrices whose ij-th entry is 1 and all others 0. Then per definition,

any $Z \in \mathcal{Z} \otimes B(\mathbb{C}^{n^k}, \mathbb{C}^{n^k})$ is of the form $\sum_{ij} z_{ij} \otimes e_{ij}$, which we identify with $\sum_{ij} z_{ij} \cdot e_{ij}$. We use the symbol $\triangleq$ to denote this identification. In particular, we have for $Z \in \mathcal{Z} \otimes B(\mathbb{C}^{n^k}, \mathbb{C}^{n^k})$,

$$Z_{12} = \sum_{ij} z_{ij} \otimes e_{ij} \otimes I = \left(\sum_{ij} z_{ij} \otimes e_{ij} \right) \otimes I \triangleq Z \otimes I, \tag{5.5}$$

$$Z_{13} = \sum_{ij} z_{ij} \otimes I \otimes e_{ij} \triangleq \sum_{ij} z_{ij} \cdot (\delta_{i_2 i} \delta_{j_2 j} \delta_{i_1 j_2})_{i_1 i_2, j_1 j_2} = I \otimes Z. \tag{5.6}$$

Since the canonical fiber functors $F_{Q(G)}, F_{Q(H)}$ are the identity on morphisms, we get that for all $T \in \mathrm{Mor}(\mathcal{U}_G^{\otimes k}, \mathcal{U}_G^{\otimes \ell})$ we have $E(T) = T$, and for all generators M_i of Q we have $F(M_i) = \Phi(M_i)$. We also get $F(A_G) = \Phi(A_G) = A_H$. An argument similar to Lemma 5.27 then shows that $F(T) = \Phi(T)$ for all $T \in \mathrm{Mor}(\mathcal{U}_G^{\otimes k}, \mathcal{U}_G^{\otimes \ell})$.

We can now apply Theorem 5.39 to the functors E and F. We let $Z^{(k)} := X^{\mathcal{U}_G^{\otimes k}}$. Then property (i) of Theorem 5.39 simplifies under the above observations and the identification of $\mathcal{Z} \otimes B(E(\mathcal{U}_G^{\otimes k}), F(\mathcal{U}_G^{\otimes k}))$ with $\mathcal{Z}^{n^k \times n^k}$ to

$$\Phi(T) Z^{(k)} = Z^{(\ell)} T,$$

as desired. By property (c) of $F^{H,G}$, we get that E_2 and F_2 are the identity. Thus, using Equations (5.5) and (5.6), property (ii) simplifies to

$$Z^{(k)} \otimes Z^{(\ell)} = (Z^{(k)} \otimes I)(I \otimes Z^{(\ell)}) \triangleq Z_{12}^{(k)} Z_{13}^{(\ell)} \triangleq Z^{\mathcal{U}_G^{\otimes k} \otimes \mathcal{U}_G^{\otimes \ell}} = Z^{\mathcal{U}_G^{\otimes (k+\ell)}} = Z^{(k+\ell)}.$$

It remains to show that the entries z_{ij} of $Z^{(1)}$ satisfy $z_{ij}^* = z_{ij}$. By Lemma 5.40, $R = \bar{R} = M^{2,0}$ solves the conjugate equations for $\mathcal{U}_G \in \mathrm{Rep}(Q(G))$. Thus by property (iii) of Theorem 5.39, we get

$$(Z^{(1)} \otimes I)^* M^{2,0} = (I \otimes Z^{(1)}) M^{2,0},$$

and we have

$$(Z^{(1)} \otimes I)^* M^{2,0} = \left(\sum_{ij} z_{ij}^* e_{ji} \otimes I \right) M^{2,0} = \sum_{ijk} z_{ij}^* (e_{ji} e_k \otimes e_k) = \sum_{ijk} z_{ij}^* (\delta_{ik} e_j \otimes e_k) = \sum_{ij} z_{ij}^* (e_j \otimes e_i)$$

and

$$(I \otimes Z^{(1)}) M^{2,0} = \left(\sum_{ij} I \otimes z_{ij} e_{ij} \right) M^{2,0} = \sum_{ijk} z_{ij} (e_k \otimes e_{ij} e_k) = \sum_{ijk} z_{ij} (e_k \otimes \delta_{jk} e_i) = \sum_{ij} z_{ij} (e_j \otimes e_i).$$

Hence $\sum_{ij} z_{ij} (e_j \otimes e_i) = \sum_{ij} z_{ij}^* (e_j \otimes e_i)$, which implies $z_{ij}^* = z_{ij}$ for all $i, j \in [n]$. □

Finally, we are able to prove the converse of Lemma 5.29.

Corollary 5.42. *Let Q be an easy quantum group and let G and H be graphs. If there exists a weak isomorphism Φ between $Q(G)$ and $Q(H)$, then there exists a quantum orthogonal matrix $\mathcal{U}$ with $\mathcal{U} A_G = A_H \mathcal{U}$, such that $\mathcal{U}^{\otimes k} M = M \mathcal{U}^{\otimes \ell}$ for all $M \in C^Q(k, \ell)$.*

Proof. By Theorem 5.41, there exist a C*-algebra $\mathcal{Z}$ and matrices $Z^{(k)} \in \mathcal{Z}^{n^k \times n^k}$ for all $k \in \mathbb{N}$, such that

(a) $Z^{(\ell)} T = \Phi(T) Z^{(k)}$ for all $T \in C_G^Q(\ell, k)$,

(b) $Z^{(k)} \otimes Z^{(\ell)} = Z^{(k+\ell)}$,

(c) the entries z_{ij} of $Z^{(1)}$ satisfy $z_{ij}^* = z_{ij}$.

Set $\mathcal{U} = Z^{(1)}$. Since $Z^{(1)}$ is unitary, (c) implies that it is orthogonal.

Claim 1. *For all $k \in \mathbb{N}$, $\mathcal{U}^{\otimes k} = Z^{(k)}$.*

Proof. First suppose $k = 0$. We know that $Z^{(0)}$ is a 1×1 matrix and thus of the form (λ) where $\lambda \in \mathcal{Z}$. Moreover, by (b) we have $\lambda Z^{(1)} = Z^{(1)} \otimes Z^{(0)} = Z^{(1)}$. Since $Z^{(1)}$ is unitary, multiplying both sides by $(Z^{(1)})^*$ yields $\lambda I = I$, which implies $\lambda = \mathbf{1}$. Consequently, $Z^0 = (\mathbf{1}) = \mathcal{U}^{\otimes 0}$. By induction we then have

$$Z^{(k+1)} = Z^{(k)} \otimes Z^{(1)} = \mathcal{U}^{\otimes k} \otimes \mathcal{U} = \mathcal{U}^{\otimes (k+1)}. \qquad \blacksquare$$

Finally suppose $M \in C^Q(\ell, k) \subseteq C_G^Q(\ell, k)$. Then by (a) we have

$$\mathcal{U}^{\otimes \ell} M = Z^{(\ell)} M = \Phi(M) Z^{(k)} = M Z^{(k)} = M \mathcal{U}^{\otimes k}.$$

Since $A_G \in C_G^Q(1,1)$, we analogously get $\mathcal{U} A_G = A_H \mathcal{U}$ as desired. □

5.5 A Characterisation of Easy Quantum Groups and their Homomorphism Indistinguishability Relations

In the previous two sections we have established the following homomorphism indistinguishability characterisation.

Theorem 5.22. *Let Q be an easy quantum group with generators $M_1, \dots, M_m$ and let $\mathcal{F}$ be the set of its intertwiner graphs. Let G and H be graphs. Then the following are equivalent.*

(*i*) $G \cong_{\mathcal{F}} H$.

(*ii*) *There exists a weak isomorphism between $Q(G)$ and $Q(H)$.*

(*iii*) *There exists a quantum orthogonal matrix $\mathcal{U}$ with $\mathcal{U}^{\otimes k} M = M \mathcal{U}^{\otimes \ell}$ for all $M \in C^Q(k, \ell)$, such that $\mathcal{U} A_G = A_H \mathcal{U}$.*

(*iv*) *There exists a quantum orthogonal matrix $\mathcal{U}$ with $\mathcal{U}^{\otimes k} M_i = M_i \mathcal{U}^{\otimes \ell}$ for all $i \in [m]$ and $M_i \in C^Q(\ell, k)$, such that $\mathcal{U} A_G = A_H \mathcal{U}$.*

Proof. We proved (i) ⇒ (ii) in Lemma 5.26, (ii) ⇒ (i) in Lemma 5.28, and (ii) ⇒ (iii) in Corollary 5.42. The implication (iii) ⇒ (iv) is trivial, and (iv) ⇒ (ii) is proved in Lemma 5.29. □

We know by Theorem 3.59 that essentially all information about the fundamental representation of a compact matrix quantum group Q is encoded in its intertwiners. Thus, property (iii) of the theorem suggests that $\mathcal{U}$ and the fundamental representation of Q share a lot of structure. The remainder of this chapter is dedicated to making this idea more precise.

5.5.1 More quantum matrices

Our first observation is that the partitions used in the classification of easy quantum groups, Theorem 3.65, in fact correspond to very concrete properties of the fundamental representations.

Lemma 5.43. *Let $\mathcal{Z}$ be a C*-algebra and $\mathcal{U} \in \mathcal{Z}^{n \times n}$. Then $\mathcal{U}$ has intertwiner*

(*i*) $\sqcap$ *iff $\mathcal{U}$ is orthogonal,*

(*ii*) $\times\!\!\times$ *iff the entries of $\mathcal{U}$ commute,*

(*iii*) $\uparrow$ *iff $\mathcal{U}$ has row sum $\mathbf{1}$,*

(*iv*) $\downarrow$ *iff* $\mathcal{U}$ *has column sum* $\mathbf{1}$,

(*v*) $\uparrow \otimes \uparrow$ *iff the product of any two row sums of* $\mathcal{U}$ *is* $\mathbf{1}$,

(*vi*) $\downarrow \otimes \downarrow$ *iff the product of any two column sums of* $\mathcal{U}$ *is* $\mathbf{1}$,

(*vii*) $\curlywedge$ *iff the entries* u_{ij} *of* $\mathcal{U}$ *satisfy* $u_{ik}u_{jk} = \delta_{ij}u_{ik}$ *for all* $i, j, k \in [n]$,

(*viii*) $\curlyvee$ *iff the entries* u_{ij} *of* $\mathcal{U}$ *satisfy* $u_{ki}u_{kj} = 0$ *and* $u_{ik}u_{jk} = 0$ *for all* $i, j, k \in [n]$ *with* $i \neq j$.

Proof. We first show (i). Expanding $\mathcal{U}^{\otimes 2}M^{2,0} = M^{2,0}(\mathbf{1})$ yields

$$\sum_i \mathcal{U}e_i \otimes \mathcal{U}e_i = \sum_i \mathbf{1}(e_i \otimes e_i). \tag{5.7}$$

Now if $\mathcal{U}$ satisfies this equation, then by the completeness relation and linearity, $\mathcal{U}$ must map orthonormal bases to orthonormal bases. This means that $\mathcal{U}$ unitary. Moreover, we have

$$(\mathcal{U}\mathcal{U}^\mathsf{T})_{ij} = \sum_k u_{ik}u_{jk} = \sum_k (\mathcal{U}e_k)_i \cdot (\mathcal{U}e_k)_j = \Big(\sum_k \mathcal{U}e_k \otimes \mathcal{U}e_k\Big)_{i\cdot j} = (M^{2,0})_{i\cdot j}\mathbf{1} = \delta_{ij}\mathbf{1}.$$

Thus $\mathcal{U}^* = \mathcal{U}^\mathsf{T} = \mathcal{U}^{-1}$, hence $\mathcal{U}$ is orthogonal. Conversely, if $\mathcal{U}$ is orthogonal then it must map orthonormal bases to orthonormal bases. The completeness relation then implies that Equation (5.7) holds.

The remaining parts can all be proven along the same lines, by expanding the intertwiner equation and then comparing the entries of the resulting matrices. We show this representatively for case (ii). Let u_{*i} and u_{i*} denote the i-th column and row vector of $\mathcal{U}$ respectively. $\mathcal{U}^{\otimes 2}S = S\mathcal{U}^{\otimes 2}$ is equivalent to

$$\mathcal{U}^{\otimes 2}\Big(\sum_{ij}(e_j \otimes e_i) \cdot (e_i \otimes e_j)^\mathsf{T}\Big) = \Big(\sum_{ij}(e_j \otimes e_i) \cdot (e_i \otimes e_j)^\mathsf{T}\Big)\mathcal{U}^{\otimes 2}.$$

Expanding this yields

$$\sum_{ij}(u_{*j} \otimes u_{*i}) \cdot (e_i \otimes e_j)^\mathsf{T} = \sum_{ij}(e_j \otimes e_i) \cdot (u_{i*} \otimes u_{j*}),$$

and entrywise comparison gives

$$\begin{aligned}
u_{i_1j_2} \cdot u_{i_2j_1} &= \sum_{ij} u_{i_1j} \cdot u_{i_2i} \cdot \delta_{ij_1} \cdot \delta_{jj_2} \\
&= \Big(\sum_{ij}(u_{*j} \otimes u_{*i}) \cdot (e_i \otimes e_j)^\mathsf{T}\Big)_{i_1i_2,j_1j_2} \\
&= \Big(\sum_{ij}(e_j \otimes e_i) \cdot (u_{i*} \otimes u_{j*})\Big)_{i_1i_2,j_1j_2} \\
&= \sum_{ij} \delta_{i_1j} \cdot \delta_{i_2i} \cdot u_{ij_1} \cdot u_{jj_2} \\
&= u_{i_2j_1} \cdot u_{i_1j_2}
\end{aligned}$$

for all $i_1, i_2, j_1, j_2 \in [n]$, as desired. □

Now recall Definition 3.42 of quantum orthogonal- and quantum permutation matrices. The previous lemma shows that we can rephrase their definition in terms of intertwiners.

Corollary 5.44. *Let $\mathcal{Z}$ be a C*-algebra. A matrix $\mathcal{U} \in \mathcal{Z}^{n\times n}$ is a*

(*i*) *quantum orthogonal matrix iff it has intertwiner* $\sqcap$,

(*ii*) *quantum permutation matrix iff it has intertwiners* $\sqcap, \uparrow, \pitchfork$.

Proof. (i) follows immediately from Lemma 5.43. For (ii), we first note that if $\mathcal{U}$ has $\uparrow$ as an intertwiner then it also has $\downarrow$ as an intertwiner, so the row and column sums are $\mathbf{1}$. It remains to show that the condition $u_{ik}u_{jk} = \delta_{ij}u_{ik}$ for all $i, j, k \in [n]$ is equivalent in this case to the weaker condition $u_{ij}^2 = u_{ij}^* = u_{ij}$ that we required in Definition 3.42.

To see this, suppose that $u_{ij}^2 = u_{ij}^* = u_{ij}$ and $\sum_\ell u_{\ell k} = \mathbf{1}$. Then

$$u_{jk} = u_{jk}\mathbf{1}u_{jk} = u_{jk}\Big(\sum_\ell u_{\ell k}\Big)u_{jk} = u_{jk} + \sum_{\ell\neq j} u_{jk}u_{\ell k}u_{jk},$$

and thus $\sum_{\ell\neq j} u_{jk}u_{\ell k}u_{jk} = 0$. Now note that all the terms $u_{jk}u_{\ell k}u_{jk} = (u_{\ell k}u_{jk})^*u_{\ell k}u_{jk}$ are positive. Since the sum of positive elements is positive again, we also have that $-u_{jk}u_{ik}u_{jk} = \sum_{\ell\neq i,j} u_{jk}u_{\ell k}u_{jk}$ is positive for all $k \in [n]$. Thus for $i \neq j$, both $u_{jk}u_{ik}u_{jk}$ and $-u_{jk}u_{ik}u_{jk}$ are positive, which is only possible if $u_{jk}u_{ik}u_{jk} = 0$. But we then have

$$\|u_{ik}u_{jk}\|^2 = \|(u_{ik}u_{jk})^*u_{ik}u_{jk}\| = \|u_{jk}u_{ik}u_{jk}\| = 0.$$

It follows that $u_{ik}u_{jk} = 0$ for $i \neq j$. Moreover, per assumption we have $u_{ik}u_{ik} = u_{ik}$, so $u_{ik}u_{jk} = \delta_{ij}u_{ik}$ as desired. □

Through their intertwiners, we can also define quantum monomial, quantum signed permutation, quantum bistochastic, and quantum signed bistochastic matrices.

Definition 5.45. Let $\mathcal{Z}$ be a C*-algebra. A non-zero matrix $\mathcal{U} \in \mathcal{Z}^{n\times n}$ is called a

(i) *quantum monomial orthogonal matrix* if it has intertwiners $\sqcap$, $\barwedge$,

(ii) *quantum signed permutation matrix* if it has intertwiners $\sqcap, \uparrow\otimes\uparrow$, $\barwedge$,

(iii) *quantum bistochastic matrix* if it has intertwiners $\sqcap, \uparrow$,

(iv) *quantum signed bistochastic matrix* if it has intertwiners $\sqcap, \uparrow\otimes\uparrow$.

In particular, the fundamental representations of H_n^+, $S_n'^+$, B_n^+, and $B_n^{\#+}$ are quantum orthogonal monomial, quantum signed permutation, quantum bistochastic, and quantum signed bistochastic matrices respectively. In the same way as for quantum orthogonal and quantum permutation matrices, we recover the notions of monomial orthogonal, signed permutation, bistochastic, and signed bistochastic matrices when the C*-algebra is $\mathbb{C}$.

Lemma 5.46. *Over $\mathbb{C}$, the following notions coincide*

(*i*) *monomial orthogonal matrices and quantum monomial orthogonal matrices,*

(*ii*) *signed permutation matrices and quantum signed permutation matrices,*

(*iii*) *bistochastic matrices and quantum bistochastic matrices,*

(*iv*) *signed bistochastic matrices and quantum signed bistochastic matrices.*

Proof. (i) Recall that monomial matrices are matrices with the same non-zero pattern as a permutation matrix, whose non-zero values are arbitrary complex numbers. Using Lemma 5.43, it is straightforward to verify that every monomial orthogonal matrix has $\sqcap$, $\sqcup\!\!\sqcap$ as intertwiners and is thus a quantum monomial orthogonal matrix. Suppose then $\mathcal{U} = (u_{ij})$ is quantum monomial orthogonal matrix over $\mathbb{C}$. Since $\mathcal{U}$ has intertwiner $\sqcap$, it is orthogonal and in particular has at least one non-zero entry per row and column. Now suppose $u_{ik} \neq 0, u_{jk} \neq 0$ for $i \neq j$. Since $u_{ik}, u_{jk} \in \mathbb{C}$, this implies $u_{ik}u_{jk} \neq 0$ contradicting—by Lemma 5.43—the fact that $\mathcal{U}$ has $\sqcup\!\!\sqcap$ as an intertwiner. The case for $u_{ki} \neq 0, u_{kj} \neq 0$ is analogous. Consequently, $\mathcal{U}$ is orthogonal and has exactly one non-zero entry per row and column, making it a monomial orthogonal matrix.

(ii) It is again straightforward to verify that every signed permutation matrix is a quantum signed permutation matrix. Let $\mathcal{U}$ be a quantum signed permutation matrix over $\mathbb{C}$. The same argument as for (i) shows that $\mathcal{U}$ is a monomial orthogonal matrix. It thus suffices to show that the non-zero entries are all 1 or all -1. Since $\uparrow \otimes \uparrow$ is an intertwiner, Lemma 5.43 implies that the product of any two row sums must equal 1. This implies that

$$1 = \Big(\sum_{\ell} u_{i\ell}\Big)\Big(\sum_{\ell} u_{i\ell}\Big) = u_{ij}^2$$

for all i and some j. In particular, all non-zero entries must be ± 1. Moreover, if there existed entries $u_{ij} = 1$ and $u_{k\ell} = -1$, then

$$\Big(\sum_{\ell} u_{i\ell}\Big)\Big(\sum_{\ell} u_{k\ell}\Big) = -1,$$

contrary to our observation. It follows that $\mathcal{U}$ is a signed permutation matrix.

(iii) Follows immediately from Lemma 5.43.

(iv) It is easy to see that every signed bistochastic matrix is a quantum signed bistochastic matrix. Suppose then $\mathcal{U}$ is quantum signed bistochastic matrix over $\mathbb{C}$. Then $\mathcal{U}$ is orthogonal because it has $\sqcap$ as an intertwiner. Moreover, the intertwiner $\uparrow \otimes \uparrow$ guarantees that the product of any two row sums equals 1. Along similar lines as in (ii), this implies that every row sum must be ± 1, and in fact all row sums must have the same value. Analogously, the intertwiner $\downarrow \otimes \downarrow$ guarantees the same for the column sums. Finally, the rows and columns must sum to the same value, otherwise

$$\sum_{ij} u_{ij} = \sum_{i}\sum_{j} u_{ij} = \sum_{j}\Big(-\sum_{i} u_{ij}\Big) = -\sum_{ij} u_{ij},$$

contradicting the fact that the sum of entries of $\mathcal{U} \in \mathbb{C}^{n\times n}$ is $\pm n$. It follows that $\mathcal{U}$ is a signed bistochastic matrix. □

An interesting situation again arises in the context of the quantum complexly modified bistochastic group, where the intertwiner spaces $\langle \times, \uparrow \otimes \uparrow \rangle$ and $\langle \times, |\backslash| \rangle$ coincide in the classical/commutative, but not in the quantum case. For this reason, we define quantum complexly signed bistochastic matrices seperately.

Definition 5.47. Let $\mathcal{Z}$ be a C*-algebra. A matrix $\mathcal{U} \in \mathcal{Z}^{n\times n}$ is called a *quantum complexly signed bistochastic matrix* if it has intertwiners $\sqcap$, $|\backslash|$.

In particular, the fundamental representation of the quantum complexly modified bistochastic group $B_n'^{+}$ is a quantum complexly signed bistochastic matrix. Since $\mathbb{C}$ is commutative, we still recover signed bistochastic matrices when $\mathcal{Z} = \mathbb{C}$.

Lemma 5.48. *Over $\mathbb{C}$, the notions of quantum complexly signed bistochastic matrices, quantum signed bistochastic matrices, and signed bistochastic matrices coincide.*

Proof. Since $\mathbb{C}$ is commutative, any quantum (complexly) signed bistochastic matrix over $\mathbb{C}$ has $\times$ as an intertwiner. Since $\mathbin{|\!\!\backslash\!\!|} \in \langle \times, \uparrow \otimes \uparrow \rangle$, every quantum signed bistochastic matrix has $\mathbin{|\!\!\backslash\!\!|}$ as an intertwiner and is thus a quantum complexly signed bistochastic matrix. Since $\uparrow \otimes \uparrow \in \langle \times, \mathbin{|\!\!\backslash\!\!|} \rangle$, every quantum complexly signed bistochastic matrix has $\uparrow \otimes \uparrow$ as an intertwiner and is thus a quantum signed bistochastic matrix. The statement now follows from Lemma 5.46. □

Finally, for each of the quantum matrices above we obtain a version with commuting entries by additionally requiring that $\times$ be an intertwiner. In the context of compact matrix quantum groups, this intertwiner is precisely what makes a quantum group classical, and we can prove a similar result for quantum matrices.

Lemma 5.49. *Let $\mathcal{Z}$ be a C*-algebra, and $\mathcal{U} \in \mathcal{Z}^{n\times n}$ be a non-zero matrix with commuting entries. Then there exists a non-zero matrix $U \in \mathbb{C}^{n\times n}$, such that $U^{\otimes k}T = T'U^{\otimes \ell}$ for all $T, T' \in \mathbb{C}^{n^k \times n^\ell}$ with $\mathcal{U}^{\otimes k}T = T'\mathcal{U}^{\otimes \ell}$.*

Proof. Let $\mathcal{Z}'$ be the C*-subalgebra of $\mathcal{Z}$ generated by the entries u_{ij} of $\mathcal{U}$. Then $\mathcal{Z}'$ is commutative and thus by Gelfand-Naimark isomorphic to the function algebra X of some finite set X. More precisely, X is the set of non-zero *-homomorphisms $\chi\colon \mathcal{Z}' \to \mathbb{C}$ and there exists an isomorphism $\phi\colon \mathcal{Z}' \to C(X)$ mapping $z \in \mathcal{Z}'$ to

$$\tilde{z}\colon X \to \mathbb{C}, \chi \mapsto \chi(z).$$

We now define a mapping $\rho\colon X \to \mathbb{C}^{n\times n}$ by $\rho(\chi)_{ij} = \phi(u_{ij})(\chi) = \chi(u_{ij})$. Since $\mathcal{Z}'$ is non-trivial, there exists a $\chi \in X$. Set $U = \rho(\chi)$ and let $T, T' \in \mathbb{C}^{n^k\times n^\ell}$ such that $\mathcal{U}^{\otimes k}T = T'\mathcal{U}^{\otimes \ell}$ for $k, \ell \geq 1$. Then since χ is an algebra homomorphism, we have

$$\begin{aligned}
\left(U^{\otimes k}T\right)_{i_1\dots i_k, j_1\dots j_\ell} &= \sum_{r_1\dots r_k} \left(\prod_s U_{i_s r_s}\right) T_{r_1\dots r_k, j_1\dots j_\ell} \\
&= \sum_{r_1\dots r_k} \left(\prod_s \chi(u_{i_s r_s})\right) T_{r_1\dots r_k, j_1\dots j_\ell} \\
&= \chi\left(\sum_{r_1\dots r_k} \left(\prod_s u_{i_s r_s}\right) T_{r_1\dots r_k, j_1\dots j_\ell}\right) \\
&= \chi\left(\left(\mathcal{U}^{\otimes k}T\right)_{i_1\dots i_k, j_1\dots j_\ell}\right) \\
&= \chi\left(\left(T'\mathcal{U}^{\otimes \ell}\right)_{i_1\dots i_k, j_1\dots j_\ell}\right) \\
&= \chi\left(\sum_{r_1\dots r_\ell} T'_{i_1\dots i_k, r_1\dots r_\ell} \prod_s u_{r_s j_s}\right) \\
&= \sum_{r_1\dots r_\ell} T'_{i_1\dots i_k, r_1\dots r_\ell} \prod_s U_{r_s j_s} \\
&= \left(T'U^{\otimes \ell}\right)_{i_1\dots r_k, j_1\dots j_\ell}.
\end{aligned}$$

The case for $k + \ell \leq 1$ follows from the observation that $\chi(\mathbf{1}) = 1$. Indeed, let $z \in \mathcal{Z}'$, $z \neq \mathbf{1}$, $z \neq 0$. Since χ is a non-zero *-homomorphism, we have $\chi(z) = \chi(z \cdot \mathbf{1}) = \chi(z) \cdot \chi(\mathbf{1})$ where $\chi(z) \neq 0$. Since $\chi(z), \chi(\mathbf{1}) \in \mathbb{C}$, $\chi(\mathbf{1})$ must be 1. An analogous argument as above then shows that for $T \in \mathbb{C}^{1\times n^\ell}$, $\mathcal{U}^{\otimes 0}T = (\mathbf{1})T = T'\mathcal{U}^{\otimes \ell}$ implies $U^{\otimes 0}T = (1)T = T'U^{\otimes \ell}$; for $T \in \mathbb{C}^{n^k\times 1}$, $\mathcal{U}^{\otimes k}T = T'\mathcal{U}^{\otimes 0} = T'(\mathbf{1})$ implies $U^{\otimes k}T = T'(1) = T'U^{\otimes 0}$; and for $T \in \mathbb{C}^{1\times 1}$, $\mathcal{U}^{\otimes 0}T = (\mathbf{1})T = T'(\mathbf{1}) = \mathcal{U}^{\otimes 0}$ implies $U^{\otimes 0}T = (1)T = T'(1) = T'U^{\otimes 0}$. This concludes the proof. □

This means that we can essentially identify quantum matrices with commuting entries with classical matrices over $\mathbb{C}$ that have the same intertwiners. In particular, Lemmas 5.43 and 5.46 apply: For instance, if there exists a quantum orthogonal matrix $\mathcal{U}$ with commuting entries and $\mathcal{U}A_G = A_H\mathcal{U}$, then there exists a corresponding orthogonal matrix U with $UA_G = A_HU$.

5.5.2 Homomorphism Indistinguishability

We have seen that in the cases of classical- and liberated quantum groups, the matrices in Theorem 5.22 either correspond to classical matrices or can be given meaningful names. Together with our results of Section 5.2, where we investigated the intertwiner graphs of the different easy quantum groups, we can state various corollaries of Theorem 5.22 in a more readable form.

The first result is a new characterisation of homomorphism indistinguishability over the class of all graphs, which according to Lovász is equivalent to classical isomorphism.

Corollary 5.50. *Let G and H be graphs. Then the following are equivalent.*

(*i*) *G and H are homomorphism-indistinguishable over the class of all graphs.*

(*ii*) *There exists a permutation matrix U such that $UA_G = A_HU$.*

(*iii*) *There exists a signed permutation matrix U such that $UA_G = A_HU$.*

(*iv*) *There exists an orthogonal monomial matrix U such that $UA_G = A_HU$.*

Moreover, we recover Mančinska and Roberson's characterisation of quantum isomorphism, avoiding the detour through quantum orbits inherent to the original proof. We also obtain two additional, equivalent characterisations that mirror the classical case.

Corollary 5.51. *Let G and H be graphs. Then the following are equivalent.*

(*i*) *G and H are homomorphism-indistinguishable over the class of all planar graphs.*

(*ii*) *There exists a quantum permutation matrix $\mathcal{U}$ such that $\mathcal{U}A_G = A_H\mathcal{U}$.*

(*iii*) *There exists a quantum signed permutation matrix $\mathcal{U}$ such that $\mathcal{U}A_G = A_H\mathcal{U}$.*

(*iv*) *There exists a quantum orthogonal monomial matrix $\mathcal{U}$ such that $\mathcal{U}A_G = A_H\mathcal{U}$.*

Homomorphism indistinguishability over the class of all cycles is well-known to be equivalent to cospectrality, and hence the existence of an orthogonal matrix U with $UA_G = A_HU$. We recover this result and extend it by a quantum version.

Corollary 5.52. *Let G and H be graphs. Then the following are equivalent.*

(*i*) *G and H are homomorphism-indistinguishable over the class of all cycles.*

(*ii*) *There exists an orthogonal matrix U such that $UA_G = A_HU$.*

(*iii*) *There exists a quantum orthogonal matrix $\mathcal{U}$ such that $\mathcal{U}A_G = A_H\mathcal{U}$.*

Finally, we recover and extend the characterisation of homomorphism indistinguishability over the class of all paths and cycles—noted by Seppelt [28, Corollary 3.3.3][6, Theorem 3] to correspond to the existence of a bistochastic matrix U with $UA_G = A_HU$—by the following criteria.

Corollary 5.53. *Let G and H be graphs. Then the following are equivalent.*

(*i*) *G and H are homomorphism-indistinguishable over the class of all paths and cycles.*

(*ii*) *There exists a bistochastic matrix U such that $UA_G = A_H U$.*

(*iii*) *There exists a signed bistochastic matrix U such that $UA_G = A_H U$.*

(*iv*) *There exists a quantum bistochastic matrix $\mathcal{U}$ such that $\mathcal{U}A_G = A_H\mathcal{U}$.*

(*v*) *There exists a quantum signed bistochastic matrix $\mathcal{U}$ such that $\mathcal{U}A_G =_H \mathcal{U}$.*

(*vi*) *There exists a quantum complexly signed bistochastic matrix $\mathcal{U}$ such that $\mathcal{U}A_G = A_H\mathcal{U}$.*

6 Conclusion

In this thesis, we explored in-depth the relation between compact quantum groups and homomorphism indistinguishability. We began by introducing the theory of compact quantum groups; focussing specifically on easy quantum groups, whose intertwiners can be defined in terms of two-sorted partitions. We then retraced the proof of Mančinska and Roberson's foundational result that quantum isomorphism is equivalent to homomorphism indistinguishability over planar graphs. Their result first established a connection between compact quantum groups and homomorphism indistinguishability by noting that the intertwiners of an easy quantum group can be characterised as homomorphism tensors counting homomorphisms from edgeless bilabelled graphs.

Based on this connection, we then proved Theorem 5.22, which characterises the feasability of the matrix equation $\mathcal{U}A_G = A_H\mathcal{U}$, where A_G, A_H are adjacency matrices of graphs G and H, and $\mathcal{U}$ is a matrix indeterminate with entries in some C*-algebra satisfying the relations $\mathcal{U}^{\otimes k}M_i = M_i\mathcal{U}^{\otimes \ell}$ for intertwiners M_i of some easy quantum group, as being equivalent to homomorphism indistinguishability over the unlabelled intertwiner graphs generated by the corresponding bilabelled graphs $\mathbf{M}_i$ and the bilabelled edge $\mathbf{A}$. In particular, the theorem generalises Mančinska and Robersons characterisation of quantum isomorphism, in which case $\mathcal{U}$ is a quantum permutation matrix, and the generated unlabelled intertwiner graphs are precisely the planar graphs.

The result is particularly nice because it joins a now quite extensive list of homomorphism indistinguishability relations characterising the feasability of matrix equations of the form $XA_G = A_HX$. Salient examples of such relations include the existence of an orthogonal matrix X, being equivalent to homomorphism indistinguishability over all cycles, and the existence of a permutation matrix X, being equivalent to homomorphism indistinguishability over all graphs. In fact, we have shown that the conditions $\mathcal{U}^{\otimes k}M_i = M_i\mathcal{U}^{\otimes \ell}$ very naturally define well-known classes of matrices and corresponding quantum relaxations. This allowed us to state some consequences of Theorem 5.22 in a form more closely resembling these existing homomorphism indistinguishability results.

The Limits of Quantum Groups

Interestingly, we reprove certain known characterisations, namely those of the existence of orthogonal, permutation, and bistochastic solutions to the matrix equation $XA_G = A_HX$. This suggests to ask the question to what extent our results are exhaustive. The short answer is that we derived all homomorphism indistinguishability relations induced by easy quantum groups. However, we have seen that easy quantum groups are special instances of the more general graph-theoretic quantum groups. These are quantum groups whose intertwiners are given by homomorphism tensors of arbitrary bilabelled graphs. It turns out that our proof generalises to these more general quantum groups without much difficulty. The resulting homomorphism indistinguishability relations, however, characterise the existence of a solution $\mathcal{U}$ not only to $\mathcal{U}A_G = A_H\mathcal{U}$, but simultaneously for $\mathcal{U}^{\otimes k_i}\mathbf{K}_G^{(i)} = \mathbf{K}_H^{(i)}\mathcal{U}^{\otimes \ell_i}$—the $\mathbf{K}^{(i)}$ being the (k_i, ℓ_i)-bilabelled graphs whose homomorphism tensors span the intertwiners of the graph-theoretic quantum group in question. However, this seems like a rather arbitrary relation from a general point of view.

A more interesting class of quantum groups have been introduced by Tarrago and Weber in [31] and [32]. They introduce the *unitary easy quantum groups* which generalise the orthogonal

easy quantum groups, that we simply called easy quantum groups until now. The unitary easy quantum groups are the quantum groups whose intertwiners can be characterised in terms of *coloured* partitions, where the elements of the underlying set are either black or white. These coloured partitions correspond to linear maps in exactly the same way as uncoloured partitions, and hence can also be seen as homomorphism tensors of edgeless bilabelled graphs. Moreover, the operations of composition, tensor product, and taking adjoints are generally defined as for their uncoloured counterparts. Merely the composition requires that the bottom row of the first and the top row of the second partition additionally match in colours. In particular, the operations on bilabelled graphs and on coloured partitions coincide whenever they are defined on partitions. We can get rid of this caveat by considering bilabelled graphs with coloured labels, allowing composition only when label colours match. This, in turn, potentially restricts the set of intertwiner graphs we are able to construct. Under these definitions, our proofs seem to go through with some obvious minor changes.

As the name suggests, the fundamental representations of unitary quantum groups do not have to be orthogonal, which makes them a promising candidate to obtain homomorphism indistinguishability relations coarser than the ones derived in Chapter 5.

Multigraphs

An indispensible tool in our construction of the intertwiner graph classes of (orthogonal) easy quantum groups is the suppression of parallel edges. This, however, has the drawback that the resulting graph classes collapse into essentially four cases. Therefore, it might be enlightening to consider the classes of multigraphs generated when keeping duplicate edges. For the results to fit into our framework of homomorphism indistinguishability, however, we have to count homomorphisms from multigraphs. One option to do this is to define

$$\hom(F,G) = \sum_{f\colon V(F)\to V(G)} \prod_{uv\in E(F)} m_{E(G)}(f(u)f(v)),$$

where $m_{E(G)}\colon E(G)\to\mathbb{N}$ is the multiplicity function of the multiset of edges $E(G)$. For bilabelled graphs $(F,\mathbf{a},\mathbf{b})$ and $(G,\mathbf{a}',\mathbf{b}')$, we instead sum only over functions $f\colon V(F)\to V(G)$ with $f(\mathbf{a}) = \mathbf{a}'$ and $f(\mathbf{b}) = \mathbf{b}'$. Under this definition, the uv-entry of $\mathbf{A}_G$ is exactly equal to the number of edges between u and v, which fits the usual definition of the adjacency matrix of a multigraph. Moreover, as Mančinska and Roberson note in the conclusion of [22], allowing multiedges does not break any of the correspondence results between homomorphism tensors, linear maps, and partitions.

An interesting caveat with respect to multigraphs is also noted by Mančinska and Roberson. Namely, there is no established definition of the quantum automorphism group of a multigraph G. Proceeding as for simple graphs, by simply requiring that $\mathcal{U}A_G = A_H\mathcal{U}$, would mean that all of our proofs go through nearly unchanged. However, this might not be the natural choice. In the classical case, we usually want to capture the permutation of parallel edges in our automorphisms, while the above definition only captures the action on vertices.

On the other hand, it is not clear that trying to mirror the classical automorphism group of multigraphs is the correct approach in the first place. To illustrate this point, recall that a central step in our proof is to add the adjacency matrix A_G to the intertwiners of an easy quantum group. For classical groups, this is equivalent to requiring that $UA_G = A_GU$ for all its elements U, which for the symmetric group happens to yield the automorphism group $\mathrm{Aut}(G)$ of G. However, it is generally not clear what it means to require that the elements U of a matrix group satisfy $UA_G = A_GU$. The symmetric group is the exception, since it acts on graphs by conjugating the adjacency matrix. $UA_G = A_GU$, being equivalent to $UA_GU^{-1} = A_G$, thus simply says that the adjacency matrix has to be invariant under the group action, which leaves precisely the permutations that are automorphisms. For other matrix groups, this conjugation does not define a group action, since UA_GU^{-1} is generally not an adjacency matrix anymore. Depending

on whether there is a natural interpretation of the subgroup of a matrix group defined by the relation $UA_G = A_G U$, simply adding A_G as an intertwiner might ultimately prove to be a more natural choice. In fact, whether such an interpretation exists is an interesting question in its own right.

Finding Linking Algebras

As recent research has shown [36], results like Mančinska and Roberson's characterisation of quantum isomorphism can generally be proven using more elementary means. This suggests to revisit the proof that the weak isomorphism of two graph instantiations $Q(G)$ and $Q(H)$ implies the existence of a linking algebra $\mathcal{Z}$ and a matrix $\mathcal{U}$ with entries in $\mathcal{Z}$ satisfying $\mathcal{U}A_G = A_H\mathcal{U}$ and $\mathcal{U}^{\otimes k}M_i = M_i\mathcal{U}^{\otimes \ell}$ for the intertwiners M_i of the underlying easy quantum group Q. The current proof obtains the linking algebra $\mathcal{Z}$ from a so called Bigalois extension of a certain Hopf *-algebra associated to the quantum group, which exists whenever $Q(G)$ and $Q(H)$ are monoidally equivalent. This is quite heavy machinery, so it might be worthwile to investigate more elementary ways of obtaining such a linking algebra.

The most obvious candidate is the underlying C*-algebra of the graph instantiation $Q(G \cup H)$. This is because the fundamental representation $\mathcal{V}$ of $Q(G \cup H) = (\mathcal{Z}, \mathcal{V}, 2n)$ already yields a matrix $\mathcal{U}$ with $\mathcal{U}A_G = A_H\mathcal{U}$. Indeed, we may assume that $A_{G\cup H} = A_G \oplus A_H$, and hence per definition

$$\mathcal{V}A_{G\cup H} = \begin{pmatrix} \mathcal{V}_1 & \mathcal{V}_2 \\ \mathcal{V}_3 & \mathcal{V}_4 \end{pmatrix} \begin{pmatrix} A_G & 0 \\ 0 & A_H \end{pmatrix} = \begin{pmatrix} A_G & 0 \\ 0 & A_H \end{pmatrix} \begin{pmatrix} \mathcal{V}_1 & \mathcal{V}_2 \\ \mathcal{V}_3 & \mathcal{V}_4 \end{pmatrix} = A_{G\cup H}\mathcal{V}.$$

In particular this means $\mathcal{V}_3 A_G = A_H \mathcal{V}_3$, where $\mathcal{V}_3 \in \mathcal{Z}^{n\times n}$. Obviously, this argument has to fail for other intertwiners, since it does not depend on the graphs G and H. An example for this is $M^{1,0} = \sum_i e_i$. If $M^{1,0}$ is an intertwiner of Q, then we have

$$\mathcal{V}M^{(2n),1,0} = \begin{pmatrix} \mathcal{V}_1 & \mathcal{V}_2 \\ \mathcal{V}_3 & \mathcal{V}_4 \end{pmatrix} \begin{pmatrix} M^{(n),1,0} \\ M^{(n),1,0} \end{pmatrix} = M^{(2n),1,0}(\mathbf{1}),$$

which is equivalent to the rows of $\mathcal{V}$ summing to $\mathbf{1}$. However, this may hold even without the row sums of $\mathcal{V}_3$ being $\mathbf{1}$. Hence we cannot conclude that $\mathcal{V}_3 M^{(n),1,0} = M^{(n),1,0}(\mathbf{1})$ without further assumptions. In general, it is not clear at all how the structure of $Q(G \cup H)$ depends on the weak isomorphism of $Q(G)$ and $Q(H)$. Beyond simplifying the proof, further inquiry in this direction may shed even more light on the role of homomorphism counts in the representation theory of graph-theoretic quantum groups.

> Well, so long, mister. Thanks for the ride, the three cigarettes and for not laughing at my theories on life.
>
> *John Garfield, The Postman Always Rings Twice*

Bibliography

[1] Philip W Anderson. "More Is Different: Broken symmetry and the nature of the hierarchical structure of science." In: *Science* 177.4047 (1972), pp. 393–396.

[2] Albert Atserias, Laura Mančinska, David E Roberson, Robert Šámal, Simone Severini, and Antonios Varvitsiotis. "Quantum and non-signalling graph isomorphisms". In: *Journal of Combinatorial Theory, Series B* 136 (2019), pp. 289–328. DOI: 10.1016/j.jctb.2018.11.002.

[3] Teodor Banica and Roland Speicher. "Liberation of orthogonal Lie groups". In: *Advances in Mathematics* 222.4 (2009), pp. 1461–1501. DOI: 10.1016/j.aim.2009.06.009.

[4] Bruce Blackadar. *Operator algebras: theory of C*-algebras and von Neumann algebras*. Encyclopaedia of mathematical sciences, Operator algebras and non-commutative geometry v. 122. 3. Berlin; New York: Springer, 2006. DOI: 10.1007/3-540-28517-2.

[5] Bruce Blackadar. "Shape theory for C*-algebras". In: *Mathematica Scandinavica* (1985), pp. 249–275. DOI: 10.7146/math.scand.a-12100.

[6] Edwin Robert van Dam, Willem H Haemers, and Jack H Koolen. "Cospectral graphs and the generalized adjacency matrix". In: *Linear Algebra and its Applications* 423.1 (2007), pp. 33–41. DOI: 10.1016/j.laa.2006.07.017.

[7] Anuj Dawar, Tomáš Jakl, and Luca Reggio. "Lovász-type theorems and game comonads". In: *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE. 2021, pp. 1–13. DOI: 10.1109/LICS52264.2021.9470609.

[8] Holger Dell, Martin Grohe, and Gaurav Rattan. "Lovász Meets Weisfeiler and Leman". In: *45th International Colloquium on Automata, Languages, and Programming (ICALP 2018)*. Vol. 107. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018, 40:1–40:14. DOI: 10.4230/LIPIcs.ICALP.2018.40.

[9] Holger Dell, Martin Grohe, and Gaurav Rattan. "Lovász Meets Weisfeiler and Leman". 2018. DOI: 10.48550/arXiv.1802.08876.

[10] Zdeněk Dvořák. "On recognizing graphs by numbers of homomorphisms". In: *Journal of Graph Theory* 64.4 (2010), pp. 330–342. DOI: 10.1002/jgt.20461.

[11] Eva Fluck, Tim Seppelt, and Gian Luca Spitzer. "Going Deep and Going Wide: Counting Logic and Homomorphism Indistinguishability over Graphs of Bounded Treedepth and Treewidth". In: *32nd EACSL Annual Conference on Computer Science Logic (CSL 2024)*. Vol. 288. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024, 27:1–27:17. DOI: 10.4230/LIPIcs.CSL.2024.27.

[12] Jonathan James Gleason. *THE C*-ALGEBRAIC FORMALISM OF QUANTUM MECHANICS*. 2009.

[13] Martin Grohe. "Counting Bounded Tree Depth Homomorphisms". In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. New York, NY, USA: Association for Computing Machinery, July 2020, pp. 507–520. DOI: 10.1145/3373718.3394739.

[14] Martin Grohe, Gaurav Rattan, and Tim Seppelt. "Homomorphism tensors and linear equations". 2021. DOI: 10.48550/arXiv.2111.11313.

[15] Martin Grohe, Gaurav Rattan, and Tim Seppelt. "Homomorphism Tensors and Linear Equations". In: *49th International Colloquium on Automata, Languages, and Programming (ICALP 2022)*. Ed. by Mikołaj Bojańczyk, Emanuela Merelli, and David P. Woodruff. Vol. 229. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 70:1–70:20. DOI: 10.4230/LIPIcs.ICALP.2022.70.

[16] Daniel Gromada. "Group-theoretical graph categories". In: *Journal of Algebraic Combinatorics* 55.2 (2022), pp. 591–627. DOI: 10.1007/s10801-021-01063-5.

[17] F. William Lawvere. "Diagonal arguments and cartesian closed categories". In: *Category Theory, Homology Theory and their Applications II*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1969, pp. 134–145. DOI: 10.1007/BFb0080769.

[18] László Lovász. "Operations with structures". In: *Acta Mathematica Hungarica* 18.3-4 (1967), pp. 321–328. DOI: 10.1007/bf02280291.

[19] Martino Lupini, Laura Mančinska, and David E Roberson. "Nonlocal games and quantum permutation groups". In: *Journal of Functional Analysis* 279.5 (2020), p. 108592. DOI: 10.1016/j.jfa.2020.108592.

[20] Laura Maaßen, Gerhard Hiß, Amaury Freslon, and Moritz Weber. "Representation categories of compact matrix quantum groups". PhD thesis. RWTH Aachen University, 2021. DOI: 10.18154/RWTH-2021-06610.

[21] Ann Maes and Alfons Van Daele. "Notes on Compact Quantum Groups". Mar. 1998. DOI: 10.48550/arXiv.math/9803122.

[22] Laura Mančinska and David E Roberson. "Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs". In: *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE. 2020, pp. 661–672. DOI: 10.1109/focs46700.2020.00067.

[23] Yoàv Montacute and Nihil Shah. "The pebble-relation comonad in finite model theory". In: *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*. 2022, pp. 1–11. DOI: 10.1145/3531130.3533335.

[24] James R Munkres. *Topology*. Vol. 7. Springer, 2000.

[25] Sergey Neshveyev and Lars Tuset. *Compact quantum groups and their representation categories*. Collection SMF 20. Paris: Société mathématique de France, 2013.

[26] Sven Raum and Moritz Weber. "The Full Classification of Orthogonal Easy Quantum Groups". In: *Communications in Mathematical Physics* 341.3 (Feb. 2016), pp. 751–779. DOI: 10.1007/s00220-015-2537-z.

[27] David E Roberson and Tim Seppelt. "Lasserre hierarchy for graph isomorphism and homomorphism indistinguishability". In: *TheoretiCS* 3 (2024). DOI: https://doi.org/10.46298/theoretics.24.20.

[28] Tim Seppelt. "Homomorphism Indistinguishability". PhD thesis. RWTH Aachen University, 2024.

[29] Tim Seppelt. "Logical equivalences, homomorphism indistinguishability, and forbidden minors". In: *Information and Computation* (2024), p. 105224. DOI: 10.1016/j.ic.2024.105224.

[30] Roland Speicher. "Free probability and non-commutative symmetries". In: *Quantum Symmetries: Metabief, France 2014* (2017), pp. 5–35. DOI: 10.1007/978-3-319-63206-3_2.

[31] Pierre Tarrago and Moritz Weber. "The classification of tensor categories of two-colored noncrossing partitions". In: *Journal of Combinatorial Theory, Series A* 154 (2018), pp. 464–506. DOI: 10.1016/j.jcta.2017.09.003.

[32] Pierre Tarrago and Moritz Weber. "Unitary easy quantum groups: the free case and the group case". In: *International Mathematics Research Notices* 2017.18 (2017), pp. 5710–5750. DOI: 10.1093/imrn/rnw185.

[33] S. L. Woronowicz. "Compact matrix pseudogroups". In: *Communications in Mathematical Physics* 111.4 (Dec. 1987), pp. 613–665. DOI: 10.1007/BF01219077.

[34] S. L. Woronowicz. "Tannaka-Krein duality for compact matrix pseudogroups. TwistedSU(N) groups". In: *Inventiones mathematicae* 93.1 (Feb. 1988), pp. 35–76. DOI: 10.1007/BF01393687.

[35] Stanisław L Woronowicz. "Compact quantum groups". In: *Symétries quantiques (Les Houches, 1995)* 845.884 (1998), p. 98.

[36] Ben Young. "The Converse of the Real Orthogonal Holant Theorem". Sept. 2024. DOI: 10.48550/arXiv.2409.06911.